Accurate Approximation Formulae for Evaluating Barrier Stock Options with Discrete Dividends and the Application in Credit Risk Valuation

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Abstract

To price the stock options with discrete dividend payout reasonably and consistently, the stock price falls due to dividend payout must be faithfully modeled. However, this will significantly increase the mathematical difficulty since the post-dividend stock price process, the stock price process after the price falls due to dividend payout, no longer follows the lognormal diffusion process. Analytical pricing formulae are hard to be derived even for the simplest vanilla options. This paper approximates the discrete dividend payout by a stochastic continuous dividend yield, so the post dividend stock price process can be approximated by another lognormally-diffusive stock process with a stochastic continuous payout ratio up to the exdividend date. Accurate approximation analytical pricing formulae for barrier options are derived by repeatedly applying the reflection principle. Besides, our formulae can be applied to extend the applicability of the first passage model— a branch of structural credit risk model. The stock price falls due to the dividend payout in the option pricing problem is analog to selling the firm's asset to finance the loan repayment or dividend payout in the first passage model. Thus our formulae can evaluate vulnerable bonds or the equity values given that the firm's future loan/dividend payments are known.

Keywords: barrier option, pricing, first-passage model

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1 Introduction

Black and Scholes (1973) arrive at their ground-breaking option pricing formula for non-dividend-paying stocks. Their option pricing model is extended to evaluate the credit risk of a defaultable firm by assuming that the firm defaults when its firm value fails to meet the debt obligation at maturity. Thus both equities and the corporate debts can be viewed as contingent claims of the firm value, and their values can be evaluated by the aforementioned Black-Scholes option pricing formula (see Merton, 1974). To deal with the dividend payout problem, Merton (1973) extends Black-Scholes formula by assuming that the stock pays a fixed continuous dividend yield. This assumption is used in the credit risk evaluation problem by allowing the firm to sell a fixed ratio of its asset continuously to finance the loan repayment or dividend payout (see Kim et al., 1993; Leland, 1994). However, most dividends and coupon payments are paid discretely rather than continuously. Pricing stock options with discrete dividend payout seems to be first investigated in Black (1975). This discrete payout setting is analog to the setting that allows the firm to discretely sell its asset to finance the loan repayment or dividend payout under the credit risk evaluation problem. Although much financial literature alternatively assumes that the firm is restricted from selling its asset (see Leland, 1994) or is allowed to sell its asset continuously at a fixed rate (see Kim et al., 1993; Leland, 1994; Leland and Toft, 1996), it is not the only— or even the typical— situation in the real world financial markets. For example, British Petroleum Plc. sold its asset to finance the spill fund demanded by the U.S. President Obama.[1] Recent news also report that many companies, like Anglo American Plc. and Potash Corp. of Saskatchewan Inc., sold their asset to meet the required dividend payments.[2] Although this discrete payment setting might be more realistic, it incurs significant mathematical difficulty since the stock price process (or the firm's value process) becomes much more complicated (see Lando, 2004).

Pricing stock options with discrete dividend payout has drawn a lot of attention in the literature. Frishling (2002) shows that the underlying stock price processes are usually modeled in three following different ways. **Model 1** suggests that the stock price minus the present value of future dividends over the life of the option follows the lognormal diffusion process (see Roll, 1977; Geske, 1979). **Model 2** suggests that the stock price plus the forward values of the dividends paid from today up to option maturity follows a lognormal diffusion process (see Heath and Jarrow (1988) and Musiela and Rutkowski (1997)). **Model**

[1]See "http://online.wsj.com/article/SB10001424052748704862404575350830340543798.html" for the news entitled as "BP Won't Issue New Equity to Cover Spill Costs."

[2]See http://www.businessweek.com/news/2010-02-17/anglo-may-resume-dividend-after-asset-sales-analysts-say.html for the news entitled as "Anglo May Resume Dividend After Asset Sales, Analysts Say" and http://fxnonstop.com/index.php/component/content/article/42555-myart26206 for the news entitled as "Potash Weighs Asset Sales for Special Dividend".

3 suggests that the stock price falls with the amount of dividend paid at the exdividend date, and follows the lognormal diffusion process between two exdividend dates. Frishling (2002) argues that these three models are incompatible with each other and generate very different prices. In addition, Frishling (2002); Bender and Vorst (2001); Bos and Vandermark (2002) argue that only Model 3 can reflect the reality and generate consistent option prices. Except the aforementioned three models, Chiras and Manaster (1978) suggests that the discrete dividends can be transformed into a fixed continuous dividend yield. The stock option can then be analytically solved by Merton's formula (see Merton, 1973). But Dai and Lyuu (2009) show that the pricing results of their approach can deviate significantly from those generated by **Model 3**. The aforementioned observations suggest that the credit risk evaluation problem could be significantly mispriced if the aforementioned approaches (except **Model 3**) are adopted.

On the other hand, pricing under **Model 3** is mathematical intractable since the post-dividend stock price process, the stock price process after the price fall due to dividend payout, is no longer log-normally distributed. Bender and Vorst (2001), Bos and Vandermark (2002), Vellekoop and Nieuwenhuis (2006), Dai and Lyuu (2009), and Dai (2009) provide approximating analytical pricing formulae or efficient numerical methods for pricing vanilla options. But no announced papers derive analytical pricing formulae for pricing barrier stock options with discrete dividend payout.

A barrier option is a popular exotic option whose payoff depends on whether the path of the underlying stock has reached a certain predetermined price level called barrier. The study of pricing barrier options is of special interesting since this problem is dual to the problem of credit risk evaluation under the first passage model— a credit risk model that models the evolution of the firm value and forces the firm to default if its value is below a certain predefined default boundary.[3] Reiner and Rubinstein (1991) derive analytical pricing formula for the barrier option given the condition that the underlying stock pays no dividend or fixed continuous dividend yield. Thus the process of the stock return can be expressed as a drifted Brownian motion and the joint density of the extreme stock price over the option life and the stock price at the option maturity date can therefore be derived by taking advantages of the reflection principle and the Girsanov's theorem. By using the risk neutral variation technique, the pricing formulae can be derived with the aforementioned joint density function. Note that Reiner and Rubinstein (1991) approach can not be directly extended to price barrier options with discrete dividend under **Model 3**, or to evaluate the equity or the corporate debt values of a defaultable firm with discrete loan or dividend payout. In addition, deal with the discrete payout with the aforementioned models except

[3] Note that the roles played by the stock price and the barrier in the barrier option pricing problem are analog to the roles played by the firm value and the default boundary in the first passage model.

Model 3 can produce unreasonable pricing results (see Frishling, 2002). Besides, Gaudenzi and Zanette (2009) develop a tree model to address this pricing problem. The impact of discrete dividend is heuristically estimated by the linear interpolation method to avoid the combinatorial explosion problem due to non-recombining property of a bushy tree (see Dai, 2009). However, it seems that their pricing results oscillate drastically due to nonlinearity error problem (see Figlewski and Gao, 1999).

The major contribution of this paper is to derive accurate approximating analytical formulae for pricing barrier options with underlying stock paying discrete dividend. Numerical results suggest that our approximation pricing formulae provide accurate option pricing results. As a byproduct, our option pricing formulae can be applied to evaluate the equity or the bond values of a defaultable firm with discrete payout. Numerical results also suggest that our formulae can explicitly show how the firm's payout to finance a debt's repayment influence the firm's financial status and the credit qualities of other outstanding debts.

To brief our approach, we first assume that the stock price $S(t)$ at time t under the risk-neutral probability is given by

$$S(t) = S(0)e^{\mu t+\sigma W(t)}, \tag{1}$$

where $\mu \equiv r - 0.5\sigma^2$, r denotes the annual risk-free interest rate, σ denotes the volatility, and $W(t)$ denotes the standard Brownian motion. Under **Model 3**, the stock pays dividend $c_1, c_2, c_3, \ldots$ at exdividend dates t_1, t_2, t_3, respectively, where $t_1 < t_2 < t_3 \ldots$. At the exdividend date t_i, the stock price falls by the amount γc_i, where the constant γ can be less than 1 to reflect the effect of tax on dividend income. In this paper, we set $\gamma = 1$ for simplicity, and any other constant $\gamma < 1$ poses no difficulties for modifying our pricing formulae. The process of the stock return prior to exdividend date t_1 can be expressed by the drifted Brownian motion: $\mu t + \sigma W(t)$ as described in Eq. (1). However, the stock price at any time t between the exdividend dates t_1 and t_2 is

$$S(t) = \left(S(0)e^{\mu t+\sigma W(t_1)} - c_1\right) e^{\mu(t-t_1)+\sigma(W(t)-W(t_1))}, \tag{2}$$

and the stock return is no longer a drifted Brownian motion. To derive analytical pricing formulae, we construct another lognormal diffusion process to approximate the price process in Eq. (2). By borrowing the idea in Dai and Lyuu (2009), the effect of stock price fall c_1 at exdividend date t_1 can be simulated by the price-fall effect contributed by a stochastic continuous dividend yield q_1 paid from time 0 to time t_1 as follows:

$$S(t_1) = S(0)e^{\mu t_1+\sigma W(t_1)} - c_1 \equiv S(0)e^{(\mu-q_1)t_1+\sigma W(t_1)}. \tag{3}$$

Thus the stock price at any time t between the two exdividend dates t_1 and t_2 can be reexpressed as

$$S(t) = S(t_1)e^{\mu(t-t_1)+\sigma(W(t)-W(t_1))} = S(0)e^{\mu t-q_1 t_1+\sigma W(t)}. \tag{4}$$

Since q_1 in Eq. (3) can be approximated solved by the first-order Taylor expansion as a affine function of $W(t_1)$, the process of the stock return between the exdividend dates t_1 and t_2, $\mu t - q_1 t_1 + \sigma W(t)$, can be approximated by a drifted Brownian motion. Let the option maturity $T < t_2$ for simplicity. The joint distribution of the extreme stock price over the time interval $[0, t_1)$ (or $[t_1, T]$) and the stock price at time t_1 (or T) can be solved by applying the reflection principle and Girsanov's theorem to the drifted Brownian motion $\mu t + \sigma W(t)$ (the approximated drifted Brownian motion of $\mu t - q_1 t_1 + \sigma W(t)$). The pricing formulae can then be derived by applying the risk-neutral valuation method on these two joint distributions. Our approach can be extended to multiple dividend dates by repeating the aforemention steps to derive approximated stock return process between any two adjacent exdividend dates.

The paper is organized as follows. Section 2 introduces required financial and mathematical background knowledge. Section 3 derives mathematical properties useful for later pricing formulae derivation. Our approximation pricing formula is then derived in section 4. Experimental results given in section 5 verify the accuracy of our pricing formulae. Section 6 concludes the paper.

2 Preliminaries

Barrier Options and the First Passage Model

Assume that a barrier stock option with strike K initiates at time 0 and matures at time T. The payoff of a up-and-out option at maturity is as follows:

$$\texttt{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\max} < B \\ 0 & \text{if } S_{\max} \geq B \end{cases},$$

where $(A)^+$ denotes $\max(A, 0)$, $S_{\max}$ denotes the maximum underlying stock price between time 0 to time T, B denotes the barrier, and θ equals 1 for call options and -1 for put options. Similarly, the payoff of a down-and-out option at maturity is as follows:

$$\texttt{payoff} = \begin{cases} (\theta S(T) - \theta K)^+ & \text{if } S_{\min} < B \\ 0 & \text{if } S_{\min} \geq B \end{cases},$$

where $S_{\min}$ denotes the minimum stock price between time 0 to time T. For simplicity, our paper will focus on up-and-out call option and the extensions to other barrier options are straight forward.

The same mathematical settings can be used to model the first passage model by redefining the symbol B as the default boundary, S as the firm value, T as the debt maturity, and K as the debt obligation at the debt maturity. The firm value process $S(t)$ is assumed to follow Eq. (1)∼(4), where σ denotes the volatility of the firm value and c_i denotes the loan repayment or dividend payout financed by selling the firm's asset at time t_i. The firm defaults once its value falls below the default boundary prior to maturity date or can not meet the debt obligation at maturity date. Thus the equity value can be evaluated as a down-and-out call option on the firm value and each debt issued by the firm can be priced by treating it as a contingent claim of the firm value.

The payoff of a up-and-out call depends on whether the stock price process has ever risen above the barrier over the life of this option. The stock price process has ever risen above the barrier during the time interval $[0, \tau]$ if and only if the maximum stock price during the time interval $[0, \tau]$ is greater than the barrier. The following theorem, derived from the reflection principle and Girsanov's theorem (see Shreve, 2007), can be applied to describe the joint density of the stock price at time τ and the maximum stock price during the time interval $[0, \tau]$.

Theorem 2.1 *Let* $\tilde{W}(t) = \alpha t + W(t)$ *be a Brownian motion with a drift term* αt *and* $\tilde{M}(\tau) = \max_{0\leq t\leq \tau} \tilde{W}(t)$ *be its maximum value over a certain time interval* $[0, \tau]$*. The joint density function of* $(\tilde{M}(\tau), \tilde{W}(\tau))$ *is given by*

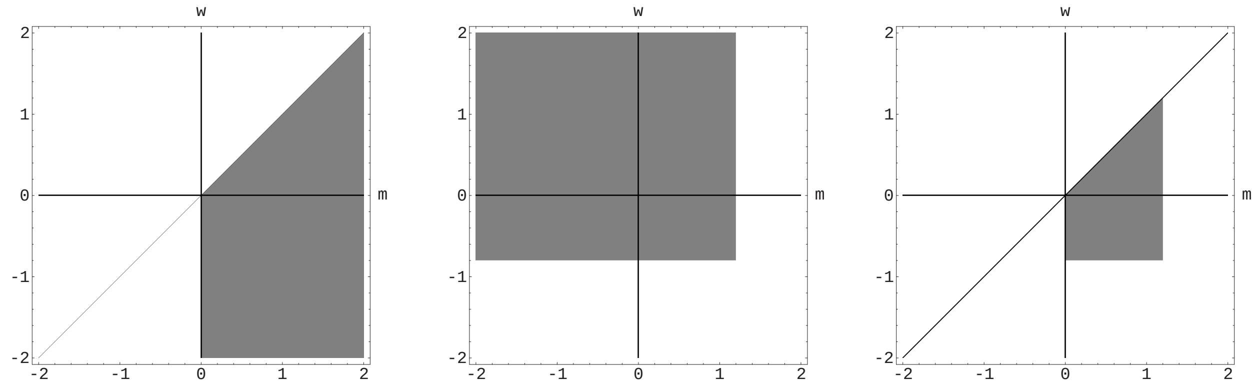
$$f_{\tilde{M}(\tau),\tilde{W}(\tau)}(m, w) = \begin{cases} \frac{2(2m-w)}{\tau\sqrt{2\pi\tau}} e^{\alpha w - \frac{1}{2}\alpha^2\tau - \frac{1}{2\tau}(2m-w)^2} & \text{if } m \geq w^+ \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

The set of points (m, w) *that make density values non-zero, also known as the support of a density, is illustrated in Fig. 1 (a).*

Reiner and Rubinstein (1991) derive analytical formulae for barrier options without discrete dividend payout by Theorem 2.1. A detail explanation of their derivation is given below since the derivation of our formula takes advantage of their method. Define the stock return in Eq.(1), $\mu t + \sigma W(t) \equiv \sigma\hat{W}(t)$, where the drifted Brownian motion $\hat{W}(t) \equiv \mu t/\sigma + W(t)$. Define the maximum value of the Brownian motion $\hat{M}(\tau)$ as $\max_{0\leq t\leq\tau} \hat{W}(t)$. Thus the value of a up-and-out call option C can be derived as follows:

$$\begin{aligned} C &= e^{-rT} E\left\{(S(T) - K)^+ 1_{\left\{\max\limits_{0\leq t\leq T} S(t)<B\right\}}\right\} \\ &= e^{-rT} E\left\{(S(0)e^{\sigma\hat{W}(T)} - K) 1_{\{S(0)e^{\sigma\hat{W}(T)}\geq K, S(0)e^{\sigma\hat{M}(T)}<B\}}\right\} \\ &= e^{-rT} E\left\{(S(0)e^{\sigma\hat{W}(T)} - K) 1_{\{\hat{W}(T)\geq k, \hat{M}(T)<b\}}\right\}, \end{aligned} \tag{6}$$

where k and b in Eq. (6) stand for $\frac{1}{\sigma}\ln\frac{K}{S(0)}$ and $\frac{1}{\sigma}\ln\frac{B}{S(0)}$, respectively. By substituting Eq.



(a) Support of Eq. (5) (b) Integral Domain of Eq. (7) (c) Taking Intersection

Figure 1: Domain of Double Integral in Eq. (8)

The shadow area in Panel (a) denotes the support of the density function of $f_{\tilde{M}(T),\tilde{W}(T)}$ in Eq. (5), i.e., a set of points (m, w) that make $f_{\tilde{M}(T),\tilde{W}(T)}(m, w)$ non-zero. Panel (b) denotes the domain of integral in Eq. (7), which is also the support of the indicator function of Eq. (6). Panel (c) denotes the intersection of shadow areas in Panel (a) and (b), which is the domain of integral in Eq. (8).

(5) into Eq. (6) with $\alpha = \sigma/\mu$, we have

$$C = \int_k^\infty \int_{-\infty}^b e^{-rT}\left(S(0)e^{\sigma w} - K\right) f_{\hat{M}(T),\hat{W}(T)}(m, w)dmdw \tag{7}$$

$$= \int_k^b \int_{w^+}^b e^{-rT}\left(S(0)e^{\sigma w} - K\right) \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2} dmdw, \tag{8}$$

where the domain of integral in Eq. (7), i.e., $-\infty < m < b$ and $k \leq w < \infty$, is the support of the indicator function in Eq. (6) as illustrated in Fig. 1 (b). The domain of integral in Eq. (8) is the intersection of the support of the joint density function $f_{\hat{M}(T),\hat{W}(T)}(m, w)$ and the support of indicator function $1_{\{\hat{W}(T)\geq k, \hat{M}(T)<b\}}$ as illustrated in Fig. 1 (c).

In the double integral formula Eq. (8), since only $f_{\hat{M}(T),\hat{W}(T)}(m, w)$ contains the variable m, $\int_{w^+}^b f_{\hat{M}(T),\hat{W}(T)}(m, w)dm$ can be evaluated first by the following lemma:

Lemma 2.2

$$\int_{v^+}^\beta \frac{2(2u-v)}{\Delta\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{1}{2\Delta}(2u-v)^2} du = \frac{1}{\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{v^2}{2\Delta}} \left(1 - e^{\frac{2\beta(v-\beta)}{\Delta}}\right).$$

Proof.

$$\int_{v^+}^\beta \frac{2(2u-v)}{\Delta\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{1}{2\Delta}(2u-v)^2} du = -\frac{1}{\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{1}{2\Delta}(2u-v)^2}\Big|_{u=v^+}^{u=\beta}$$

$$= -\frac{1}{\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta}\left(e^{-\frac{(2\beta-v)^2}{2\Delta}} - e^{-\frac{v^2}{2\Delta}}\right) = \frac{1}{\sqrt{2\pi\Delta}} e^{\alpha v - \frac{1}{2}\alpha^2\Delta - \frac{v^2}{2\Delta}}\left(1 - e^{\frac{2\beta(v-\beta)}{\Delta}}\right).$$

$\square$

By applying Lemma 2.2, Eq. (8) can be rewritten as

$$\begin{aligned}
C &= e^{-rT}\int_k^b (S(0)e^{\sigma w}-K)\left(\int_{w^+}^b \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w-\frac{1}{2}\alpha^2 T-\frac{1}{2T}(2m-w)^2}dm\right)dw \\
&= e^{-rT}\int_k^b (S(0)e^{\sigma w}-K)\frac{1}{\sqrt{2\pi T}} e^{\alpha w-\frac{1}{2}\alpha^2 T-\frac{w^2}{2T}}\left(1-e^{\frac{2b(w-b)}{T}}\right)dw \\
&= e^{-rT}\int_k^b -\frac{K}{\sqrt{2\pi T}}e^{-\frac{w^2}{2T}+\alpha w-\frac{T\alpha^2}{2}}+\frac{K}{\sqrt{2\pi T}}e^{-\frac{w^2}{2T}+\alpha w-\frac{T\alpha^2}{2}+\frac{2b(w-b)}{T}} \\
&\quad +\frac{S'(0)}{\sqrt{2\pi T}}e^{-\frac{w^2}{2T}+\alpha w+\sigma w-\frac{T\alpha^2}{2}}-\frac{S'(0)}{\sqrt{2\pi T}}e^{-\frac{w^2}{2T}+\alpha w+\sigma w-\frac{T\alpha^2}{2}+\frac{2b(w-b)}{T}}dw. \qquad (9)
\end{aligned}$$

In Eq. (9), each term of the integrand is of the form $Le^{a_2w^2+a_1w+a_0}$ for some constants a_0, a_1, a_2, L. The following identity can convert the integrals of the aforementioned form into the cumulative distribution function (CDF) of the standard normal distribution:

$$\int_{-\infty}^{l} e^{a_2x^2+a_1x+a_0}dx=\sqrt{\frac{\pi}{-a_2}}e^{-\frac{a_1^2-4a_0a_2}{4a_2}}N\left(\frac{l-\mathbf{m}}{\mathbf{s}}\right), \qquad (10)$$

where $\mathbf{m}=-\frac{a_1}{2a_2}$, $\mathbf{s}=\frac{1}{\sqrt{-2a_2}}$, $N(\cdot)$ denotes the CDF of the standard normal distribution. The detailed derivation of this identity is given in Eq. (46) in Appendix A. Thus, Reiner and Rubinstein (1991) pricing formula can be derived as a linear combination of tail probability values, which can be evaluated by the CDF of the standard normal distribution.

3 Derivations of Useful Mathematical Properties

Approximate the Post-dividend Stock Price Process with a Lognormal Diffusion Process

In this section, we derive lognormal diffusion processes to approximate the stock price process between two adjacent exdividend dates under **Model 3**. Note that the stock price process before the first exdividend date is already a lognormal diffusion process (see Eq. (1)). But the stock price process after the first exdividend date t_1 illustrated in Eq. (2) is no longer lognormally distributed. Our approximated process is derived by replacing the discrete dividend c_1 paid at time t_1 with a continuous dividend q_1 paid from time 0 to time t_1 as illustrated in Eq. (3) so the approximated stock price after time t_1 can be expressed as Eq. (4). To make this process a lognormal diffusion one, q_1 is approximately solved as a linear function of $W(t_1)$ from Eq. (3) as follows:

$$\begin{aligned} S(0)e^{\mu t_1+\sigma W(t_1)} - c_1 &\approx S(0)e^{\mu t_1+\sigma W(t_1)}(1-q_1 t_1) && (11)\\ \Rightarrow -c_1 &\approx -q_1 t_1 S(0)e^{\mu t_1+\sigma W(t_1)} \\ \Rightarrow \quad q_1 &\approx \frac{c_1 e^{-\mu t_1}\cdot e^{-\sigma W(t_1)}}{t_1 S(0)} \\ \Rightarrow \quad q_1 &\approx \frac{c_1 e^{-\mu t_1}(1-\sigma W(t_1))}{t_1 S(0)}, && (12) \end{aligned}$$

where the first order Taylor expansion $e^x \approx 1 + x$ is used in Eq. (11) and (12). By substituting $k_1 \equiv \frac{c_1 e^{-\mu t_1}}{S(0)} - 1$, $q_1 \equiv \frac{(k_1-1)(1-\sigma W(t_1))}{t_1}$ into Eq.(4), we obtain the lognormal process to approximate the stock price process between time t_1 and t_2 as follows:

$$\begin{aligned} S(t) &\approx S(0)e^{\mu t-(k_1-1)(1-\sigma W(t_1))+\sigma W(t_1)+\sigma(W(t)-W(t_1))} \\ &= S(0)e^{(\mu t-k_1+1)+k_1\sigma W(t_1)+\sigma(W(t)-W(t_1))}. && (13) \end{aligned}$$

The aforementioned procedure can be repeated to find the lognormal diffusion process that approximate the stock price process between the exdividend dates t_2 and t_3. Again, we replace the discrete dividend c_2 with a continuous dividend yield q_2 paid from time t_1 to time t_2; that is,

$$\begin{aligned} S(t) &= \left(S(t_1)e^{\mu(t_2-t_1)+\sigma(W(t_2)-W(t_1))} - c_2\right) e^{\mu(t-t_2)+\sigma(W(t)-W(t_2))} \\ &= S(0)e^{(\mu-q_1)t_1+\sigma W(t_1)} e^{(\mu-q_2)(t_2-t_1)+\sigma(W(t_2)-W(t_1))} e^{\mu(t-t_2)+\sigma(W(t)-W(t_2))}. && (14) \end{aligned}$$

Note that q_2 can be approximately solved by the first-order Taylor expansion to get

$$q_2 \approx \frac{(k_2-1)\left[1-k_1\sigma W(t_1)-\sigma(W(t_2)-W(t_1))\right]}{t_2-t_1}, \tag{15}$$

where $k_2 \equiv \frac{c_2 e^{-\mu t_2+k_1-1}}{S(0)} - 1$. Therefore the stock price process between exdividend dates t_2 and t_3 can be approximated by a lognormal diffusion process by substituting Eq. (15) into Eq. (14) to get

$$S(t) \approx S(0)e^{(\mu t-k_1-k_2+2)+k_1k_2\sigma W(t_1)+k_2\sigma(W(t_2)-W(t_1))+\sigma(W(t)-W(t_2))}. \tag{16}$$

Thus the approximated stock price process $\hat{S}(t)$ used to derive the pricing formulae later is constructed by combining Eqs. (1), (13), and (16) as follows:

$$\hat{S}(t) = \begin{cases} S(0)e^{\mu t+\sigma W(t)} & 0 \le t < t_1, \\ S(0)e^{(\mu t-k_1+1)+k_1\sigma W(t_1)+\sigma(W(t)-W(t_1))} & t_1 \le t < t_2, \\ S(0)e^{(\mu t-k_1-k_2+2)+k_1k_2\sigma W(t_1)+k_2\sigma(W(t_2)-W(t_1))+\sigma(W(t)-W(t_2))} & t_2 \le t < t_3. \end{cases} \tag{17}$$

To derive the pricing formulae with Theorem 2.1, the stock return in Eq. (17) should be reexpressed in terms of drifted Brownian motions. Let $\alpha \equiv \mu/\sigma$ for convenience. First, the process $\hat{S}(t)$ for $t \in [0, t_1)$ is rewritten as

$$S(0)e^{\sigma \hat{W}(t)} \tag{18}$$

by substituting $\hat{W}(t) \equiv \alpha t + W(t)$ for $t \in [0, t_1)$ into the first line of Eq. (17). The process $\hat{S}(t)$ for $t \in [t_1, t_2)$ is rewritten from the second line of Eq. (17) as follows:

$$\begin{aligned} \hat{S}(t) &= S(0)e^{(\mu t - k_1 + 1) + k_1 \sigma (W(t_1)) + \sigma (W(t) - W(t_1))} \\ &= S(0)e^{(1-k_1)(1+\mu t_1) + k_1[\mu t_1 + \sigma W(t_1)] + [\mu(t - t_1) + \sigma(W(t) - W(t_1))]} \\ &= S^{'}(0)e^{k_1 \sigma \hat{W}(t_1) + \sigma \hat{W}_1(t - t_1)}, \end{aligned} \tag{19}$$

where the drifted Brownian motion $\hat{W}_1(t - t_1) \equiv \alpha(t - t_1) + (W(t) - W(t_1))$ for $t \in [t_1, t_2)$, and $S^{'}(0) \equiv S(0)e^{(1-k_1)(1+\mu t_1)}$. The stock price process $\hat{S}(t)$ for $t \in [t_2, t_3)$ can be further rewritten from the third line of Eq. (17) as

$$\begin{aligned} \hat{S}(t) &= S(0)e^{(\mu t - k_1 - k_2 + 2) + k_1 k_2 \sigma W(t_1) + k_2 \sigma (W(t_2) - W(t_1)) + \sigma (W(t) - W(t_2))} \\ &= S(0) \exp \{ (\mu t - k_1 - k_2 + 2) - k_1 k_2 \mu t_1 - k_2 \mu (t_2 - t_1) - \mu (t - t_2) \\ &\qquad + k_1 k_2 [\mu t_1 + \sigma (W(t_1))] \\ &\qquad + k_2 [\mu (t_2 - t_1) + \sigma (W(t_2) - W(t_1))] \\ &\qquad + [\mu (t - t_2) + \sigma (W(t) - W(t_2))] \} \\ &= S^{''}(0)e^{k_1 k_2 \sigma \hat{W}(t_1) + k_2 \sigma \hat{W}_1(t_2 - t_1) + \sigma \hat{W}_2(t - t_2)}, \end{aligned} \tag{20}$$

where $S^{''}(0) \equiv S(0)e^{(\mu t - k_1 - k_2 + 2) - k_1 k_2 \mu t_1 - k_2 \mu (t_2 - t_1) - \mu (t - t_2)}$, and $\hat{W}_2(t - t_2) \equiv \alpha(t - t_2) + (W(t) - W(t_2))$ for $t \in [t_2, t_3)$. By combining Eqs. (18), (19), and (20), the approximated stock price process $\hat{S}(t)$ can be rewritten as

$$\hat{S}(t) = \begin{cases} S(0)e^{\sigma \hat{W}(t)} & 0 \leq t < t_1, \\ S^{'}(0)e^{k_1 \sigma \hat{W}(t_1) + \sigma \hat{W}_1(t - t_1)} & t_1 \leq t < t_2, \\ S^{''}(0)e^{k_1 k_2 \sigma \hat{W}(t_1) + k_2 \sigma \hat{W}_1(t_2 - t_1) + \sigma \hat{W}_2(t - t_2)} & t_2 \leq t < t_3. \end{cases} \tag{21}$$

Note that the aforementioned method can be recursively applied to solve the approximated stock price process between arbitrary two adjacent exdividend dates.

Evaluate the Integration of Exponential Functions

Indeed, most option pricing formulae, including the formulae in this paper, can be expressed in terms of multiple integrations of an exponential function, the exponent term of which is a quadratic function of integrators. Theorem 3.1 shows that such an integration can be expressed as a CDF of multi-variate normal distribution by taking advantages of some matrix and vector calculations. For simplicity, for any matrix $\daleth$, we use $|\daleth|$, $\daleth^T$ and $\daleth^{-1}$ to denote the determinant, the transpose, and the inverse of $\daleth$. $\daleth_{i,j}$ stands for the element located at the i-th row and j-th column of $\daleth$. For any vector ν, we use ν_i to denote the i-th element of ν.

Theorem 3.1 *Let $\mathbf{x}$ and $\mathbf{B}$ be an $n \times 1$ constant vector, $\mathbf{C}$ be a constant, and $\mathbf{A}$ be an $n \times n$ symmetric invertible negative-definite constant matrix. For any general quadratic formula $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C}$, the n-variate integral for $e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}}$*

$$\int_{-\infty}^{p_n}\int_{-\infty}^{p_{n-1}}\cdots\int_{-\infty}^{p_1} e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}}d\mathbf{x} \tag{22}$$

can be expressed in terms of a CDF of a n-dimensional standard normal distribution $F_{Y_1,Y_2,\cdots,Y_n}$ with covariance matrix Σ as follows:

$$e^{\mathbf{C}'}\sqrt{\frac{\pi^n}{|-\mathbf{A}|}}F_{Y_1,Y_2,\cdots,Y_n}\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}},\frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}},\cdots,\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}},\Sigma\right), \tag{23}$$

where the vector $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n) \equiv -\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}$, $\mathbf{C}' \equiv \mathbf{C} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$, $\Sigma \equiv (-2\mathbf{SAS})^{-1}$, and S is a $n \times n$ diagonal matrix defined as

$$\mathbf{S}_{i,j} \equiv \begin{cases} \sqrt{((-2\mathbf{A})^{-1})_{i,i}} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Proof. See Appendix A. □

4 Analytical Formulae

We will first derive the approximating analytical pricing formula for the up-and-out barrier call with single discrete dividend in Sec. 4.1. Our approach can be extended to derive pricing formulae for the multi-dividend case as discussed in Sec. 4.2.

4.1 The Single-discrete-dividend Case

Since only one dividend is paid during the life of the option, the option maturity date T is longer than the first exdividend date t_1 but smaller than the second exdividend date t_2 (i.e,

$t_1 < T < t_2$). The call option value $\dot{C}$ can be evaluated by the risk-neutral valuation method as follows:

$$\dot{C} \equiv e^{-rT} E\left[(\hat{S}(T) - K) 1_{\left\{\dot{E}_1 \cap \dot{E}_2 \cap \dot{E}_3\right\}}\right], \tag{24}$$

where $\dot{E}_1, \dot{E}_2$ denote the events that the stock price process does not hit barrier B during time interval $[0, t_1)$ and $[t_1, T]$, respectively, and $\dot{E}_3$ denotes the event that stock price is greater than the strike price K at maturity date. Specifically, the three events $\dot{E}_1, \dot{E}_2, \dot{E}_3$ are defined as follows:

$$\begin{aligned}
\dot{E}_1 &\equiv \left\{\hat{S}(t) < B, \quad \forall t \in [0, t_1)\right\}, \\
\dot{E}_2 &\equiv \left\{\hat{S}(t) < B, \quad \forall t \in [t_1, T]\right\}, \\
\dot{E}_3 &\equiv \left\{\hat{S}(T) > K\right\}.
\end{aligned} \tag{25}$$

To evaluate the option value, we derive the joint density of the maximum stock prices over the time interval $[0, t_1)$ and $[t_1, T]$ and the stock price at time t_1 and T by Theorem 2.1. Let $\hat{M}(t_1) \equiv \max_{0 \le t < t_1} \hat{W}(t)$ be the maximum value of $\hat{W}(t)$ over the time interval $[0, t_1)$, and $\hat{M}_1(T - t_1) \equiv \max_{t_1 \le t \le T} \hat{W}_1(t - t_1)$ be the maximum value of $\hat{W}_1(t - t_1)$ over the time interval $[t_1, T]$. Thus the three events $\dot{E}_1, \dot{E}_2, \dot{E}_3$ can be rewritten by substituting the definition of $\hat{S}(t)$ defined in Eq. (21) into Eq. (25) to get

$$\begin{aligned}
\dot{E}_1 &= \left\{S(0) e^{\sigma \hat{M}(t_1)} < B\right\} = \left\{\hat{M}(t_1) < b\right\}, \\
\dot{E}_2 &= \left\{S^{'}(0) e^{k_1 \sigma \hat{W}(t_1) + \sigma \hat{M}_1(T - t_1)} < B\right\} = \left\{\hat{M}_1(T - t_1) < b^{'} - k_1 \hat{W}(t_1)\right\}, \\
\dot{E}_3 &= \left\{S^{'}(0) e^{k_1 \sigma \hat{W}(t_1) + \sigma \hat{W}_1(T - t_1)} > K\right\} = \left\{\hat{W}_1(T - t_1) > k^{'} - k_1 \hat{W}(t_1)\right\},
\end{aligned}$$

where b, $b^{'}$ and $k^{'}$ represent $\frac{1}{\sigma} \log \frac{B}{S(0)}$, $\frac{1}{\sigma} \log \frac{B}{S^{'}(0)}$ and $\frac{1}{\sigma} \log \frac{K}{S^{'}(0)}$ for simplicity. In addition, Theorem 2.1 says that the joint density functions $f_{\hat{M}(t_1), \hat{W}(t_1)}$ and $f_{\hat{M}_1(T-t_1), \hat{W}_1(T-t_1)}$ are given as follows:

$$\dot{f}_{\hat{M}(t_1), \hat{W}(t_1)}(m, w) = \begin{cases} \frac{2(2m - w)}{t_1 \sqrt{2\pi t_1}} e^{\alpha w - \frac{1}{2}\alpha^2 t_1 - \frac{1}{2t_1}(2m - w)^2} & \text{if } m \ge w^+, \\ 0 & \text{otherwise}, \end{cases} \tag{26}$$

$$\dot{f}_{\hat{M}_1(T-t_1), \hat{W}_1(T-t_1)}(m_1, w_1) = \begin{cases} \frac{2(2m_1 - w_1)}{(T - t_1)\sqrt{2\pi (T - t_1)}} e^{\alpha w_1 - \frac{1}{2}\alpha^2 (T - t_1) - \frac{1}{2(T - t_1)}(2m_1 - w_1)^2} & \text{if } m_1 \ge w_1^+, \\ 0 & \text{otherwise}. \end{cases} \tag{27}$$

For simplicity, from now on we will use the symbols $\dot{f}_0$ and $\dot{f}_1$ to represent $f_{\hat{M}(t_1), \hat{W}(t_1)}$ and $f_{\hat{M}_1(T-t_1), \hat{W}_1(T-t_1)}$ respectively. Since the two drifted Brownian motion $\hat{W}(t)$ for $t \in [0, t_1)$ and $\hat{W}_1(t - t_1)$ for $t \in [t_1, T]$ are independent due to the Markov property of Brownian motion, the joint density function of maximum stock prices over $[0, t_1)$ and $[t_1, T]$ and the stock prices at time t_1 and T can be calculated by directly multiplying $\dot{f}_0$ by $\dot{f}_1$. By substituting the above arguments into Eq.(24), the analytical pricing formula can be derived by the law of iterated expectation as follows:

$$\begin{aligned}
\dot{C} &= e^{-rT} E\left[E\left[(\hat{S}(T)-K)1_{\{\dot{E}_1\cap\dot{E}_2\cap\dot{E}_3\}}\middle|\hat{W}(t_1),\hat{M}(t_1)\right]\right] \\
&= e^{-rT}\int_{-\infty}^{\infty}\int_{-\infty}^{b}\int_{k'-k_1w}^{\infty}\int_{-\infty}^{b'-k_1w}\left(S'(0)e^{k_1\sigma w+\sigma w_1}-K\right)\dot{f}_1(m_1,w_1)\dot{f}_0(m,w)dm_1dw_1dmdw \quad (28)\\
&= e^{-rT}\int_{-\infty}^{b}\int_{w^+}^{b}\int_{k'-k_1w}^{b'-k_1w}\int_{w_1^+}^{b'-k_1w}\left(S'(0)e^{k_1\sigma w+\sigma w_1}-K\right)\dot{f}_1(m_1,w_1)\dot{f}_0(m,w)dm_1dw_1dmdw \quad (29)
\end{aligned}$$

where the domain of integral in Eq. (29) is obtained by mimicking the analysis in Fig. 1; it is derived by taking the intersection of the supports of $\dot{f}_1(m_1, w_1)$ and $\dot{f}_0(m_0, w_0)$ with the integral domain in Eq. (28). Since only $\dot{f}_0(m, w)$ contains m and $f_1(m_1, w_1)$ contains m_1 in the integrand in Eq. (29), $\int_{w^+}^{b}\dot{f}_0(m, w)dm$ and $\int_{w_1^+}^{b'-k_1w}\dot{f}_1(m_1, w_1)dm_1$ can be integrated separately by Lemma 2.2 as follows:

$$\begin{aligned}
\dot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{k'-k_1w}^{b'-k_1w}\left(S'(0)e^{k_1\sigma w+\sigma w_1}-K\right)\\
&\quad\cdot\left(\int_{w_1^+}^{b'-k_1w}\frac{2(2m_1-w_1)}{(T-t_1)\sqrt{2\pi(T-t_1)}}e^{\alpha w_1-\frac{1}{2}\alpha^2(T-t_1)-\frac{1}{2(T-t_1)}(2m_1-w_1)^2}dm_1\right)\\
&\quad\cdot\left(\int_{w^+}^{b}\frac{2(2m-w)}{t_1\sqrt{2\pi t_1}}e^{\alpha w-\frac{1}{2}\alpha^2t_1-\frac{1}{2t_1}(2m-w)^2}dm\right)dw_1dw. \qquad (30)
\end{aligned}$$

To eliminate the variables in the lower limit and the upper limit for the integral on w_1, we substitute $x = w_1 + k_1w$ and $y = w$ into Eq. (30) to get[4]

$$\begin{aligned}
\dot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{k'}^{b'}\left(S'(0)e^{\sigma x}-K\right) \qquad (31)\\
&\quad\cdot\frac{1}{\sqrt{2\pi(T-t_1)}}e^{\alpha(x-k_1y)-\frac{1}{2}\alpha^2(T-t_1)-\frac{(x-k_1y)^2}{2(T-t_1)}}\left(1-e^{\frac{2\left(b'-k_1y\right)\left(x-k_1y-\left(b'-k_1y\right)\right)}{T-t_1}}\right)\\
&\quad\cdot\frac{1}{\sqrt{2\pi t_1}}e^{\alpha y-\frac{1}{2}\alpha^2t_1-\frac{y^2}{2t_1}}\left(1-e^{\frac{2b(y-b)}{t_1}}\right)dxdy.\\
&= e^{-rT}\int_{-\infty}^{b}\int_{k'}^{b'}I(1)+I(2)+I(3)+I(4)+I(5)+I(6)+I(7)+I(8)dxdy, \qquad (32)
\end{aligned}$$

where the integrand $I(1)$ is defined as

$$I(1) \equiv -\frac{K}{2\pi\sqrt{(T-t_1)t_1}}e^{-\frac{y^2}{2t_1}+\alpha y+\alpha(x-yk_1)-\frac{1}{2}\alpha^2(T-t_1)-\frac{\alpha^2t_1}{2}-\frac{(x-yk_1)^2}{2(T-t_1)}},$$

[4]Note that the Jacobian determinant $\frac{\partial(w_1,w)}{\partial(x,y)} = 1$.

$I(2) \equiv -I(1)e^{\frac{2(x-b^{'})\left(b^{'}-yk_1\right)}{T-t_1}}$, $I(3) \equiv -I(1)e^{\frac{2b(y-b)}{t_1}}$, and $I(4) \equiv I(1)e^{\frac{2\left(x-b^{'}\right)\left(b^{'}-yk_1\right)}{T-t_1}+\frac{2b(y-b)}{t_1}}$. These four terms are obtained by multiplying the strike price K in Eq. (31) with the terms in the following two lines. $I(5)$, $I(6)$, $I(7)$, and $I(8)$, which are obtained by multiplying $S'(0)e^{\sigma x}$ in Eq. (31) with the terms in the following two lines, are defined as

$$I(i) = -\frac{S'(0)}{K}I(i-4)e^{\sigma x}, \qquad i = 5, \ldots, 8. \tag{33}$$

Since each exponent term of the integrands $I(1), I(2), \cdots, I(8)$ is a quadratic polynomial of integrators x and y, the double integral of each integrand can be expressed in terms of a bivariate normal CDF by the following lemma:

Corollary 4.1 *The double integral G with the following format can be expressed in terms of the CDF of a bivariate standard normal distribution F_{Y_1,Y_2} as follows:*

$$\begin{aligned} & G(p,q,a_1,a_2,a_3,a_4,a_5,a_6) \equiv \int_{-\infty}^{q}\int_{-\infty}^{p} e^{a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6}dxdy \\ = & \frac{2\pi}{\sqrt{\Delta}}\exp\left(a_6+\frac{a_2a_4a_5-a_3a_4^2-a_1a_5^2}{\Delta}\right)F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta}p+\frac{2a_3a_4-a_2a_5}{\sqrt{\Delta}}}{\sqrt{-2a_3}}, \frac{\sqrt{\Delta}q+\frac{2a_1a_5-a_2a_4}{\sqrt{\Delta}}}{\sqrt{-2a_1}}, \Sigma\right), \end{aligned} \tag{34}$$

where $\Delta \equiv 4a_1a_3 - a_2^2$, *and* $\Sigma \equiv \begin{pmatrix} 1 & \frac{a_2}{2\sqrt{a_1a_3}} \\ \frac{a_2}{2\sqrt{a_1a_3}} & 1 \end{pmatrix}$.

Proof. It can be easily derived from Theorem 3.1 and is proved in Appendix B. □

With Corollary 4.1, the double integrals of $I(1), I(2), \cdots, I(8)$ can be converted into CDFs of bivariate normal distributions. For example, the integral of $I(1)$ can be rewritten as

$$\begin{aligned} & \int_{-\infty}^{b}\int_{k'}^{b'} I(1)dxdy = \int_{-\infty}^{b}\int_{k'}^{b'} -\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}}e^{-\frac{y^2}{2t_1}+\alpha y+\alpha(x-yk_1)-\frac{1}{2}\alpha^2(T-t_1)-\frac{\alpha^2 t_1}{2}-\frac{(x-yk_1)^2}{2(T-t_1)}}dxdy \\ = & -\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}}\left(\int_{-\infty}^{b}\int_{-\infty}^{b'} e^{-\frac{1}{2(T-t_1)}x^2+\frac{k_1}{T-t_1}xy-\left(\frac{k_1^2}{2(T-t_1)}+\frac{1}{2t_1}\right)y^2+\alpha x+(\alpha-\alpha k_1)y-\frac{T\alpha^2}{2}}dxdy \right. \\ & \left. -\int_{-\infty}^{b}\int_{-\infty}^{k'} e^{-\frac{1}{2(T-t_1)}x^2+\frac{k_1}{T-t_1}xy-\left(\frac{k_1^2}{2(T-t_1)}+\frac{1}{2t_1}\right)y^2+\alpha x+(\alpha-\alpha k_1)y-\frac{T\alpha^2}{2}}dxdy\right). \end{aligned} \tag{35}$$

Define $a_1(1) \equiv -\frac{1}{2(T-t_1)}$, $a_2(1) \equiv \frac{k_1}{T-t_1}$, $a_3(1) \equiv -\left(\frac{k_1^2}{2(T-t_1)}+\frac{1}{2t_1}\right)$, $a_4(1) \equiv \alpha$, $a_5(1) \equiv (\alpha - \alpha k_1)$, $a_6(1) \equiv -\frac{T\alpha^2}{2}$ be the coefficients of x^2, xy, y^2, x, y and the constant term, respectively, of the exponential term of the integrand $I(1)$ in Eq. (35). By Corollary 4.1, Eq. (35) can be rewritten in terms of bivariate normal CDFs as follows:

$$-\frac{K}{2\pi\sqrt{(T-t_1)\,t_1}}\frac{2\pi}{\sqrt{\Delta(1)}}\exp\left(a_6(1)+\frac{a_2(1)a_4(1)a_5(1)-a_3(1)a_4(1)^2-a_1(1)a_5(1)^2}{\Delta(1)}\right)$$
$$\cdot\left[F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta(1)}b^{'}+\frac{2a_3(1)a_4(1)-a_2(1)a_5(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_3(1)}},\frac{\sqrt{\Delta(1)}b+\frac{2a_1(1)a_5(1)-a_2(1)a_4(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_1(1)}},\Sigma(1)\right)\right.$$
$$\left.-F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta(1)}k^{'}+\frac{2a_3(1)a_4(1)-a_2(1)a_5(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_3(1)}},\frac{\sqrt{\Delta(1)}b+\frac{2a_1(1)a_5(1)-a_2(1)a_4(1)}{\sqrt{\Delta(1)}}}{\sqrt{-2a_1(1)}},\Sigma(1)\right)\right], \quad (36)$$

where $\Delta(1)$, $\Sigma(1)$ are obtained by substituting $a_1(1), a_2(1), \cdots, a_6(1)$ into Corollary 4.1 as follows:

$$\Delta(1) \equiv 4a_1(1)a_3(1)-a_2(1)^2,$$
$$\Sigma(1) \equiv \begin{pmatrix} 1 & \frac{a_2(1)}{2\sqrt{a_1(1)a_3(1)}} \\ \frac{a_2(1)}{2\sqrt{a_1(1)a_3(1)}} & 1 \end{pmatrix}.$$

For convenience, we rewrite Eq. (36) as follows:

$D(1)[G(b^{'},b,a_1(1),a_2(1),a_3(1),a_4(1),a_5(1),a_6(1))-G(k^{'},b,a_1(1),a_2(1),a_3(1),a_4(1),a_5(1),a_6(1))]$,

where $D(1)\equiv -\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$, and G is defined in Eq. (34). Similarly, the double integrals for $I(2)$, $I(3)$, …, $I(8)$ in Eq. (32) can all be expressed as

$$\int_{-\infty}^{b}\int_{k^{'}}^{b^{'}} I(i)dxdy = D(i)[G(b^{'},b,a_1(i),a_2(i),a_3(i),a_4(i),a_5(i),a_6(i))$$
$$-G(k^{'},b,a_1(i),a_2(i),a_3(i),a_4(i),a_5(i),a_6(i))],$$

where $a_1(i)$, $a_2(i)$, $a_3(i)$, $a_4(i)$, $a_5(i)$, and $a_6(i)$ denotes the coefficients of x^2, xy, y^2, x, y and the constant term of the exponential term of $I(i)$. Specifically, the parameters are given by $a_1(i)=a_1(1)$, $a_3(i)=a_3(1)$, $a_2(i)=(-1)^{i+1}a_2(1)$ for $i=2,3,\cdots,8$, and $D(i)$, $a_4(i)$, $a_5(i)$, $a_6(i)$ are given by the following table:

i	$D(i)$	$a_4(i)$	$a_5(i)$	$a_6(i)$
2	$\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b^{'}}{T-t_1}+\alpha$	$\alpha+k_1\left(\frac{2b^{'}}{T-t_1}-\alpha\right)$	$-\frac{4b^{'2}+T^2\alpha^2-T\alpha^2t_1}{2T-2t_1}$
3	$\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	α	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
4	$-\frac{K}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b^{'}}{T-t_1}+\alpha$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{T-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b^{'2}}{T-t_1}$
5	$\frac{S^{'}(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\alpha+\sigma$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
6	$-\frac{S^{'}(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b^{'}}{T-t_1}+\alpha+\sigma$	$\alpha+k_1\left(\frac{2b^{'}}{T-t_1}-\alpha\right)$	$-\frac{4b^{'2}+T^2\alpha^2-T\alpha^2t_1}{2T-2t_1}$
7	$-\frac{S^{'}(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\alpha+\sigma$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
8	$\frac{S^{'}(0)}{2\pi\sqrt{(T-t_1)t_1}}$	$\frac{2b^{'}}{T-t_1}+\alpha+\sigma$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{T-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b^{'2}}{T-t_1}$

Thus, the option pricing formula in Eq. (32) can be rewritten as

$$\begin{aligned}
\dot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{k'}^{b'} I(1)+I(2)+\cdots+I(8)dxdy \\
&= e^{-rT}\sum_{i=1}^{8}\Big[D(i)[G(b',b,a_1(i),a_2(i),a_3(i),a_4(i),a_5(i),a_6(i)) \\
&\qquad -G(k',b,a_1(i),a_2(i),a_3(i),a_4(i),a_5(i),a_6(i))]\Big].
\end{aligned}$$

Note that if the upper barrier $B\to\infty$, the up-and-out call becomes a vanilla call option. Indeed, $b=\frac{1}{\sigma}\log\frac{B}{S(0)}\to\infty$ and $b'=\frac{1}{\sigma}\log\frac{B}{S'(0)}\to\infty$ and our pricing formula degenerate into an approximating pricing formula for vanilla stock call options with one discrete dividend payout derived in Dai and Lyuu (2009).

4.2 Multi-discrete-dividend Case

The above approach can be repeatedly applied to derive approximated pricing formulae for barrier stock options with multiple discrete dividend payout. For simplicity, we derive the pricing formula for the two-dividend case in this section. The extensions for three or more dividends cases are straightforward. Note that $t_1<t_2<T<t_3$ in the two-dividend case.

To evaluate the option, we need to derive the joint density function of the maximum stock prices over the time intervals $[0,t_1)$, $[t_1,t_2)$, and $[t_2,T]$ and the stock price at time T. Let $\hat{M}_1(t_2-t_1)\equiv\max_{t_1\le t<t_2}\hat{W}_1(t-t_1)$ be the maximum value of $\hat{W}_1(t)$ over the time interval $[t_1,t_2)$ and $\hat{M}_2(T-t_2)\equiv\max_{t_2\le t\le T}\hat{W}_2(t-t_2)$ be the maximum value of $\hat{W}_2(t)$ over the time interval $[t_2,T]$. The joint density function of $\hat{M}_1(t_2-t_1)$ and $\hat{W}_1(t_2-t_1)$ and the density function of $\hat{M}_2(T-t_2)$ and $\hat{W}_2(T-t_2)$ can be derived by applying Theorem 2.1 as follows:

$$f_{\hat{M}_1(t_2-t_1),\hat{W}_1(t_2-t_1)}(m_1,w_1) = \begin{cases}\frac{2(2m_1-w_1)}{(t_2-t_1)\sqrt{2\pi(t_2-t_1)}}e^{\alpha w_1-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{1}{2(t_2-t_1)}(2m_1-w_1)^2} & \text{if } m_1\ge w_1^+,\\ 0 & \text{otherwise,}\end{cases} \tag{37}$$

$$f_{\hat{M}_2(T-t_2),\hat{W}_2(T-t_2)}(m_2,w_2) = \begin{cases}\frac{2(2m_2-w_2)}{(T-t_2)\sqrt{2\pi(T-t_2)}}e^{\alpha w_2-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2(T-t_2)}(2m_2-w_2)^2} & \text{if } m_2\ge w_2^+,\\ 0 & \text{otherwise.}\end{cases} \tag{38}$$

For simplicity, we use $\ddot{f}_0$, $\ddot{f}_1$, and $\ddot{f}_2$ to represent the density functions $f_{\hat{M}(t_1),\hat{W}(t_1)}$ (see Eq. (26)), $f_{\hat{M}_1(t_2-t_1),\hat{W}_1(t_2-t_1)}$, and $f_{\hat{M}_2(T-t_2),\hat{W}_2(T-t_2)}$, respectively. Note that the drifted Brownian motions $\hat{W}(t)$ for $t\in[0,t_1)$, $\hat{W}_1(t-t_1)$ for $t\in[t_1,t_2)$, and $\hat{W}_2(t-t_2)$ for $t\in[t_2,t_3]$ are independent due to the Markov property of Brownian motion, the joint density function of maximum stock prices over $[0, t_1)$, $[t_1, t_2)$ and $[t_2, T]$ and the stock prices at time t_1, t_2, and T can be calculated by directly multiplying $\ddot{f}_0$ with $\ddot{f}_1$ and $\ddot{f}_2$.

The option value can be evaluated by the risk-neutral variation method as follows:

$$\ddot{C} \equiv e^{-rT} E\left[(\hat{S}(T) - K) 1_{\{\ddot{E}_1 \cap \ddot{E}_2 \cap \ddot{E}_3 \cap \ddot{E}_4\}}\right], \tag{39}$$

where $\ddot{E}_1$, $\ddot{E}_2$, $\ddot{E}_3$ represent the events that the stock price process does not hit the barrier B during the time interval $[0, t_1), [t_1, t_2)$ and $[t_2, T]$, respectively, and $\ddot{E}_4$ denotes the event that the stock price at maturity is greater than the strike price. Specifically, $\ddot{E}_1$, $\ddot{E}_2$, $\ddot{E}_3$, and $\ddot{E}_4$ are defined as

$$\begin{aligned}
\ddot{E}_1 &= \left\{S(0)e^{\sigma \hat{M}(t_1)} < B\right\} = \left\{\hat{M}(t_1) < b\right\}, \\
\ddot{E}_2 &= \left\{S^{'}(0)e^{k_1\sigma\hat{W}(t_1) + \sigma\hat{M}_1(t_2 - t_1)} < B\right\} = \left\{\hat{M}_1(t_2 - t_1) < b^{'} - k_1\hat{W}(t_1)\right\}, \\
\ddot{E}_3 &= \left\{S^{''}(0)e^{k_1k_2\sigma\hat{W}(t_1) + k_2\sigma\hat{W}_1(t_2 - t_1) + \sigma\hat{M}_2(T - t_2)} < B\right\} \\
&= \left\{\hat{M}_2(T - t_2) < b^{''} - k_1k_2\hat{W}(t_1) - k_2\hat{W}_1(t_2 - t_1)\right\}, \\
\ddot{E}_4 &= \left\{S^{''}(0)e^{k_1k_2\sigma\hat{W}(t_1) + k_2\sigma\hat{W}_1(t_2 - t_1) + \sigma\hat{W}_2(T - t_2)} < K\right\} \\
&= \left\{\hat{W}_2(T - t_2) < k^{''} - k_1k_2\hat{W}(t_1) - k_2\hat{W}_1(t_2 - t_1)\right\},
\end{aligned}$$

where $k^{''} \equiv \frac{1}{\sigma}\log\frac{K}{S^{''}(0)}$, and $b^{''} \equiv \frac{1}{\sigma}\log\frac{B}{S^{''}(0)}$, respectively. Thus, we can compute the pricing formula in Eq.(39) by applying the law of iterated expectation as follows:

$$\begin{aligned}
\ddot{C} &= e^{-rT} E\Big[E\Big[E\Big[(\hat{S}(T) - K) 1_{\{\ddot{E}_1 \cap \ddot{E}_2 \cap \ddot{E}_3 \cap \ddot{E}_4\}} \\
&\qquad \Big| \hat{W}(t_1), \hat{M}(t_1), \hat{W}_1(t_2 - t_1), \hat{M}_1(t_2 - t_1)\Big] \Big| \hat{W}(t_1), \hat{M}(t_1)\Big]\Big] \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{b^{'} - k_1 w} \int_{k^{''} - k_1k_2w - k_2w_1}^{\infty} \int_{-\infty}^{b^{''} - k_1k_2w - k_2w_1} \\
&\quad \left(S^{''}(0)e^{k_1k_2\sigma w + k_2\sigma w_1 + \sigma w_2} - K\right) \cdot \ddot{f}_2(m_2, w_2) \cdot \ddot{f}_1(m_1, w_1) \cdot \ddot{f}_0(m, w) dm_2 dw_2 dm_1 dw_1 dm dw \qquad (40) \\
&= e^{-rT} \int_{-\infty}^{b} \int_{w^+}^{b} \int_{-\infty}^{b^{'} - k_1 w} \int_{w_1^+}^{b^{'} - k_1 w} \int_{k^{''} - k_1k_2w - k_2w_1}^{b^{''} - k_1k_2w - k_2w_1} \int_{w_2^+}^{b^{''} - k_1k_2w - k_2w_1} \left(S^{''}(0)e^{k_1k_2\sigma w + k_2\sigma w_1 + \sigma w_2} - K\right) \\
&\quad \cdot \ddot{f}_2(m_2, w_2) \cdot \ddot{f}_1(m_1, w_1) \cdot \ddot{f}_0(m, w) dm_2 dw_2 dm_1 dw_1 dm dw, \qquad (41)
\end{aligned}$$

where the domain of integral in Eq. (41) is obtained by by taking the intersection of the supports of $\ddot{f}_2(m_2, w_2)$, $\ddot{f}_1(m_1, w_1)$, $\ddot{f}_0(m, w)$ with the integral domain in Eq. (40). Since only $\ddot{f}_2(m_2, w_2)$ contains m_2, $\ddot{f}_1(m_1, w_1)$ contains m_1, and $\ddot{f}_0(m, w)$ contains m in the integrand in Eq. (41), $\int \ddot{f}_0(m, w) dm$, $\int \ddot{f}_1(m_1, w_1) dm_1$, and $\int \ddot{f}_2(m_2, w_2) dm_2$ can be simplified by applying

Lemma 2.2 as follows:

$$\begin{aligned}
\ddot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}-k_1w}\int_{k^{''}-k_1k_2w-k_2w_1}^{b^{''}-k_1k_2w-k_2w_1}\left(S^{''}(0)e^{k_1k_2\sigma w+k_2\sigma w_1+\sigma w_2}-K\right)\\
&\quad\cdot\left(\int_{w_2^+}^{b^{''}-k_1k_2w-k_2w_1}\frac{2(2m_2-w_2)}{(T-t_2)\sqrt{2\pi(T-t_2)}}e^{\alpha w_2-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2(T-t_2)}(2m_2-w_2)^2}dm_2\right)\\
&\quad\cdot\left(\int_{w_1^+}^{b^{'}-k_1w}\frac{2(2m_1-w_1)}{(t_2-t_1)\sqrt{2\pi(t_2-t_1)}}e^{\alpha w_1-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{1}{2(t_2-t_1)}(2m_1-w_1)^2}dm_1\right)\\
&\quad\cdot\left(\int_{w^+}^{b}\frac{2(2m-w)}{t_1\sqrt{2\pi t_1}}e^{\alpha w-\frac{1}{2}\alpha^2t_1-\frac{1}{2t_1}(2m-w)^2}dm\right)dw_2dw_1dw.
\end{aligned}$$

To eliminate the variables in the lower and the upper limits for the integrals on w_1 and w_2, we substitute $x = w_2 + k_2w_1 + k_1k_2w$, $y = w_1 + k_1w$, and $z = w$ into the aforementioned formula to get[5]

$$\begin{aligned}
\ddot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}}\int_{k^{''}}^{b^{''}}\left(S^{''}(0)e^{\sigma x}-K\right)\\
&\quad\cdot\frac{1}{\sqrt{2\pi(T-t_2)}}e^{\alpha(x-k_2y)-\frac{1}{2}\alpha^2(T-t_2)-\frac{(x-k_2y)^2}{2(T-t_2)}}\left(1-e^{\frac{2\left(b^{''}-k_2y\right)\left(x-k_2y-\left(b^{''}-k_2y\right)\right)}{(T-t_2)}}\right)\\
&\quad\cdot\frac{1}{\sqrt{2\pi(t_2-t_1)}}e^{\alpha(y-k_1z)-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{(y-k_1z)^2}{2(t_2-t_1)}}\left(1-e^{\frac{2\left(b^{'}-k_1z\right)\left(y-k_1z-\left(b^{'}-k_1z\right)\right)}{(t_2-t_1)}}\right)\\
&\quad\cdot\frac{1}{\sqrt{2\pi t_1}}e^{\alpha z-\frac{1}{2}\alpha^2t_1-\frac{z^2}{2t_1}}\left(1-e^{\frac{2b(z-b)}{t_1}}\right)dxdydz = e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b^{'}}\int_{k^{''}}^{b^{''}}\sum_{i=1}^{16}J(i)dxdydz, \qquad (42)
\end{aligned}$$

where $J(1), J(2), \cdots, J(8)$ are defined in Table 1, and $J(9), J(10), \cdots, J(16)$ are defined as

$$J(i) = -\frac{S''(0)}{K}J(i-8)e^{x\sigma}, \quad i = 9, \ldots, 16.$$

Since the exponent term of each of the integrands $J(1)$, $J(2)$, $\cdots$, $J(16)$ is a quadratic form of integrators x, y, and z, the triple integral of each integrand can be expressed in terms of a trivariate normal CDF by the following corollary:

Corollary 4.2 *The triple integral with the following formats can be expressed in terms of a trivariate standard normal distribution F_{Y_1,Y_2,Y_3} as follows:*

$$\begin{aligned}
H(p,q,r,a_1,a_2,\cdots,a_{10}) &\equiv \int_{-\infty}^{r}\int_{-\infty}^{q}\int_{-\infty}^{p}e^{a_1x^2+a_2y^2+a_3z^2+a_4xy+a_5yz+a_6xz+a_7x+a_8y+a_9z+a_{10}}dxdydz\\
&= e^{C'}\sqrt{\frac{\pi^3}{|-A|}}F_{Y_1,Y_2,Y_3}\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}},\frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}},\frac{p_3-\mathbf{m}_3}{\mathbf{S}_{3,3}},\Sigma\right), \qquad (43)
\end{aligned}$$

[5]Note that the Jacobian determinant $\frac{\partial(w_2,w_1,w)}{\partial(x,y,z)}$ is 1.

Table 1: The definitions of $J(1)$, $J(2)$, …, $J(8)$.

Define ζ as $\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}$, and η as $-\frac{z^2}{2t_1}+\alpha z+\alpha(y-zk_1)+\alpha(x-yk_2)-\frac{\alpha^2 t_1}{2}-\frac{1}{2}\alpha^2(T-t_2)-\frac{1}{2}\alpha^2(t_2-t_1)-\frac{(x-yk_2)^2}{2(T-t_2)}-\frac{(y-zk_1)^2}{2(t_2-t_1)}$.

$J(1)$	$=$	$-\frac{K}{\zeta}e^{\eta}$,	$J(2)$	$=$	$\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}}$,
$J(3)$	$=$	$\frac{K}{\zeta}e^{\eta+\frac{2\left(x-b^{''}\right)\left(b^{''}-yk_2\right)}{T-t_2}}$,	$J(4)$	$=$	$-\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(x-b^{''}\right)\left(b^{''}-yk_2\right)}{T-t_2}}$,
$J(5)$	$=$	$\frac{K}{\zeta}e^{\eta+\frac{2\left(y-b^{'}\right)\left(b^{'}-zk_1\right)}{t_2-t_1}}$,	$J(6)$	$=$	$-\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(y-b^{'}\right)\left(b^{'}-zk_1\right)}{t_2-t_1}}$,
$J(7)$	$=$	$-\frac{K}{\zeta}e^{\eta+\frac{2\left(x-b^{''}\right)\left(b^{''}-yk_2\right)}{T-t_2}+\frac{2\left(y-b^{'}\right)\left(b^{'}-zk_1\right)}{t_2-t_1}}$,	$J(8)$	$=$	$\frac{K}{\zeta}e^{\eta+\frac{2b(z-b)}{t_1}+\frac{2\left(x-b^{''}\right)\left(b^{''}-yk_2\right)}{T-t_2}+\frac{2\left(y-b^{'}\right)\left(b^{'}-zk_1\right)}{t_2-t_1}}$,

where

$$A=\begin{pmatrix} a_1 & \frac{a_4}{2} & \frac{a_6}{2} \\ \frac{a_4}{2} & a_2 & \frac{a_5}{2} \\ \frac{a_6}{2} & \frac{a_5}{2} & a_3 \end{pmatrix}, B=\begin{pmatrix} a_7 \\ a_8 \\ a_9 \end{pmatrix}, \mathbf{S}=\begin{pmatrix} \mathbf{S}_{1,1} & 0 & 0 \\ 0 & \mathbf{S}_{2,2} & 0 \\ 0 & 0 & \mathbf{S}_{3,3} \end{pmatrix},$$

$\mathbf{S}_{j,j}=\sqrt{((-2A)^{-1})_{j,j}}$, $\mathbf{m}=-\frac{1}{2}A^{-1}B$, $C^{'}=a_{10}-\frac{1}{4}B^TA^{-1}B$, *and* $\Sigma=(-2\mathbf{S}A\mathbf{S})^{-1}$.

Proof. This corollary can be easily derived from Theorem 3.1. □

Let $a_1(i)$, $a_2(i)$, $a_3(i)$, $a_4(i)$, $a_5(i)$, $a_6(i)$, $a_7(i)$, $a_8(i)$, $a_9(i)$, and $a_{10}(i)$ be the coefficients of x^2, y^2, z^2, xy, yz, xz, x, y, z and the constant term, respectively, of the exponential term of the integrand $J(i)$ in Table 1. These coefficients are listed in Appendix C. The triple integrals of $J(i)$ in Eq. (42) can be expressed in terms of CDFs of trivariate normal distributions by applying Corollary 4.2 as follows:

$$\int_{-\infty}^{b}\int_{-\infty}^{b^{'}}\int_{k^{''}}^{b^{''}} J(i)dxdydz = E(i)[H(b^{''},b^{'},b,a_1(i),a_2(i),\cdots,a_{10}(i))-H(k^{''},b^{'},b,a_1(i),a_2(i),\cdots,a_{10}(i))], \tag{44}$$

where the function H is defined in Eq. (43), and the function $E(i)$ is defined as follows:

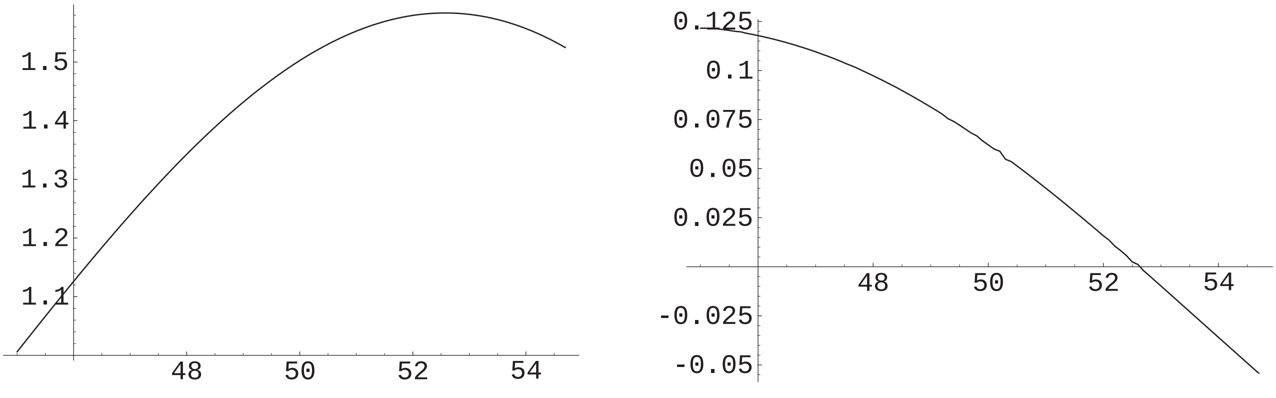
$$\begin{aligned} E(2) &= E(3)=E(5)=E(8)=\frac{K}{\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}}, \\ E(1) &= E(4)=E(6)=E(7)=-E(2), \\ E(9) &= E(12)=E(14)=E(15)=\frac{S^{''}(0)}{\sqrt{8\pi^3(T-t_2)(t_2-t_1)t_1}}, \\ E(10) &= E(11)=E(13)=E(16)=-E(9), \end{aligned}$$

By substituting Eq. (44) into Eq. (42), the option price formula for the two-dividend case is derived as follows:

$$\begin{aligned}\ddot{C} &= e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b'}\int_{k''}^{b''} J(1)+J(2)+\cdots+J(16)dxdydz \\ &= e^{-rT}\sum_{i=1}^{16}\Big[E(i)[H(b'',b',b,a_1(i),a_2(i),\cdots,a_{10}(i)) - H(k'',b',b,a_1(i),a_2(i),\cdots,a_{10}(i))]\Big].\end{aligned}$$

5 Numerical Results

Unlike most numerical pricing approaches that will generate oscillating pricing results as mentioned in Figlewski and Gao (1999) and Dai and Lyuu (2010), our approximate pricing formulae can generate smooth and stable pricing results as illustrated in Fig. 2. In panel (a), the up-and-out call option value increases with the initial stock price when the initial stock price is low. However, a higher initial stock price also increases the probability for the option to knock out (i.e., the stock price path hits the barrier), and the option values decreases with the initial stock price when the initial stock price is high. We can also observe that the delta decreases with the initial stock price in panel (b).



(a) Option Value

(b) Delta

Figure 2: Option Value and Delta

The x-axes in both panels denote the initial stock price. The risk-free rate is 3%, the volatility is 20%, the strike price is 50, the barrier is 65, and the time to maturity is 1 year. A dividend 1 is paid at 0.5 year. Panel (a) gives the plot of the up-and-out barrier call price against the initial stock price. Panel (b) gives the plot of the delta against the initial stock price.

To examine the superiority of our pricing formula, we will compare our approximation pricing formulae with other pricing schemes. In the following tables, the pricing results in `Benchmark` columns are generated by the Monte Carlo simulation with 1,000,000 paths and we use these prices as the benchmark. `Ours` denotes the approximation pricing formulae proposed in this paper. The discrete dividends paid over the life of the option can be approximated by the equivalent continuous dividend yield q (see Chiras and Manaster, 1978) defined as follows:

$$S(0)e^{-qT} = S(0) - \sum_{i=1}^{n} c_i e^{-rt_i}, \tag{45}$$

where n denotes the number of dividends paid during the life of the option. Then the discrete-dividend barrier option can be approximately priced by the barrier option pricing formula with a continuous dividend yield, and the pricing results generated by this approach are listed under the `ContDiv` columns. Besides, we can follow Model 1 (see Roll (1977)) by assuming that the process of the net-of-dividend stock price follows a lognormal diffusion process with price at time 0 defined by

$$S_N(0) \equiv S(0) - \sum_{i=1}^{n} c_i e^{-rt_i}.$$

Thus the discrete-dividend barrier option can be approximately priced by the barrier option pricing formula by replacing the initial stock price with $S_N(0)$. The prices generated by this approach are listed under the `Model1` columns in the following tables.[6]

Table 2 illustrates how the changes of the initial stock prices influence the option values and the accuracy of the aforementioned three pricing formulae. Similar as Fig. 2 (a) , the option value increases with the increment of the initial stock price when the initial price is low. The option value then decreases with the increment of the initial stock price when the initial price is high. Our pricing formula is more accurate than the other two formulae since the maximum absolute error (`MAE`) and the root-mean-squared error (`RMSE`) are lower than those for the other two formulae. In addition, the pricing errors of `Model1` are much more significant than the errors of the other two formulae. `Model1` produces very inaccurate results (the percentage of error=$\frac{0.1765}{0.1932} \approx 90\%$) when the initial stock price is high, say 64. Table 3 compares the pricing results under different amount of discrete dividend payout. It can be observed that the pricing errors of both `ContDiv` and `Model1` increases with the amount of dividend payout, while the pricing errors of `Ours` are much smaller. `MAE` and `RMSE` of `Ours` are also smaller than those of `ContDiv` and `Model1`. Table 4 illustrates the pricing results under different volatilities. Note that the value of an up-and-out call decreases as the volatility increases since a higher volatility implies a higher "knock out" probability. It can be also observed that `MAE` and `RMSE` of `Ours` are all smaller than 10^{-2}, while `MAE` and `RMSE` of both `ContDiv` and `Model1` are much more higher. Table 5 illustrates the impacts the change of exdividend date. By observing the `Benchmark` column, the benchmark value

[6]Frishling (2002) argues that Model 1 could incorrectly render a down-and-out barrier option worthless simply because the net-of-dividend stock price reaches the barrier when the present value of future dividends is big enough.

$S(0)$	`Benchmark`	`Ours`	error	`ContDiv`	error	`Model1`	error
46	1.1265	1.1260	0.0005	1.1124	0.0141	1.1336	0.0071
48	1.3456	1.3427	0.0029	1.3317	0.0139	1.3641	0.0184
50	1.5054	1.5026	0.0028	1.4952	0.0102	1.5417	0.0363
52	1.5829	1.5796	0.0033	1.5767	0.0062	1.6401	0.0572
54	1.5661	1.5571	0.0089	1.5598	0.0063	1.6422	0.0762
56	1.4389	1.4310	0.0079	1.4395	0.0005	1.5423	0.1034
58	1.2112	1.2093	0.0019	1.2228	0.0116	1.3463	0.1352
60	0.9164	0.9106	0.0059	0.9266	0.0102	1.0700	0.1536
62	0.5667	0.5602	0.0065	0.5745	0.0078	0.7358	0.1691
64	0.1932	0.1868	0.0065	0.1932	0.0000	0.3697	0.1765
`MAE`		0.0089		0.0141		0.1765	
`RMSE`		0.0054		0.0093		0.1109	

Table 2: Comparing the Effect of Changing Initial Stock Prices on Pricing Barrier Calls with Single Discrete Dividend
The initial stock price is listed in the first column, All other numerical settings are the same as those in Fig. 2. The columns "error" list the absolute pricing error between each pricing formula and the benchmark. MAE denotes the maximum absolute error and RMSE denotes the root-mean-squared error.

decreases as the time to exdividend date t_1 increases. Our formula successfully captures this phenomenon, while the other two approaches fail.

Next, we extend our comparison to the two-dividend case. The underlying stock is assumed to pay two dividends at year 0.5 and year 1 and the time to maturity is set to 1.5 years. Table 6 illustrates the impact of changing the initial stock price and Table 7 illustrates the impact of changing the amount of dividend payout. Again, `MAE` and `RMSE` of `Ours` are also smaller than those of `ContDiv` and `Model1`. Thus we verify the superiority of our pricing formulae.

Next, we move our discussion to the credit risk problem. The results for pricing the vulnerable bond with face value \$3000 is illustrated in Fig. 3, where x-axis denotes the bond maturity. We assume that the firm value is 5000, the volatility of the firm value is 25%, an exogenously given default boundary is 2400, the firm regularly pays a dividend with amount \$150 by selling its asset value per 1.5 year, and the risk-free rate is 2%. Note that the dividend payment weakens the financial status of the firm and increases the risk of the bond that are repayed after the exdividend date. It can be observed that the pricing results of our formula (marked in solid and hollow squares) catch the trend of the significant price fall of the

c_1	Benchmark	Ours	error	ContDiv	error	Model1	error
0.3	1.5759	1.5730	0.0029	1.5705	0.0054	1.5857	0.0098
0.6	1.5438	1.5435	0.0003	1.5387	0.0051	1.5680	0.0242
0.9	1.5202	1.5129	0.0073	1.5062	0.0140	1.5486	0.0283
1.2	1.4868	1.4815	0.0053	1.4729	0.0139	1.5273	0.0405
1.5	1.4478	1.4493	0.0015	1.4390	0.0088	1.5044	0.0566
1.8	1.4147	1.4163	0.0017	1.4045	0.0102	1.4798	0.0652
2.1	1.3843	1.3828	0.0015	1.3694	0.0150	1.4538	0.0695
2.4	1.3459	1.3488	0.0030	1.3338	0.0121	1.4262	0.0804
MAE		0.0073		0.0150		0.0804	
RMSE		0.0036		0.0112		0.0523	

Table 3: Comparing the Effect of Changing the Amount of Dividend on Pricing Barrier Calls with Single Discrete Dividend
All settings are the same as the settings in Table 2 except that the initial stock price is set as 50 and the dividend c_1 is listed in the first column.

vulnerable bond from 2893.77 at year 1.5 to 2885.52 at year 1.52. To keep analytical formula tractable, many credit risk papers (like Kim et al. (1993) and Leland (1994)) heuristically assume that the firm continuously sells a predetermined ratio of its asset, but the pricing results (marked in solid triangles) can not precisely reflect the change of firm's financial status due to the discrete payout. This heuristic assumption underprices (or overprices) the bond when the bond maturity is less (or greater) than the exdividend date.

6 Conclusions

Most stock dividends are paid discretely rather than continuously. Up to now, there is no announced analytical formula for pricing barrier stock options with discrete dividends, and pricing these options by the formulae for continuous dividend payout could lead to unreasonable pricing results. This paper provides accurate approximating analytical formulae for pricing barrier stock option with discrete dividend payout. Numerical results are given to confirm the superiority of our formulae to other analytical formulae. Our formula can be extended the applicability of the first passage model, a popular credit risk model. The stock price falls due to dividend payout is analog to selling the firms asset to finance the debt or dividend repayments. Thus our formulae can estimate the credit risk under the first passage model given that the firms future loan/dividend payments are known.

Volatility	`Benchmark`	`Ours`	error	`ContDiv`	error	`Model1`	error
0.1	2.0707	2.0756	0.0049	2.0552	0.0154	2.0612	0.0094
0.2	1.5054	1.5026	0.0028	1.4952	0.0102	1.5417	0.0363
0.3	0.7215	0.7167	0.0047	0.7172	0.0043	0.7534	0.0320
0.4	0.3625	0.3611	0.0014	0.3627	0.0002	0.3846	0.0221
0.5	0.2035	0.1998	0.0037	0.2013	0.0022	0.2144	0.0109
0.6	0.1205	0.1197	0.0007	0.1209	0.0004	0.1292	0.0087
0.7	0.0767	0.0764	0.0003	0.0773	0.0006	0.0827	0.0060
0.8	0.0526	0.0511	0.0014	0.0518	0.0007	0.0556	0.0030
0.9	0.0366	0.0356	0.0010	0.0361	0.0005	0.0388	0.0021
1.0	0.0255	0.0255	0.0000	0.0260	0.0005	0.0279	0.0024
MAE		0.0079		0.0929		0.1389	
RMSE		0.0042		0.0430		0.0625	

Table 4: Comparing the Effect of Changing the Volatility on Pricing Barrier Calls with Single Discrete Dividend
All numerical settings are the same as those settings in Table 2 except that the initial stock price is 50 and the volatility of the stock price is listed in the first column.

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t_1	`Benchmark`	`Ours`	error	`ContDiv`	error	`Model1`	error
0.1	1.5425	1.5378	0.0047	1.4938	0.0486	1.5408	0.0016
0.2	1.5347	1.5335	0.0012	1.4942	0.0405	1.5410	0.0063
0.3	1.5291	1.5262	0.0029	1.4945	0.0346	1.5412	0.0121
0.4	1.5236	1.5160	0.0076	1.4948	0.0287	1.5415	0.0179
0.5	1.5054	1.5026	0.0028	1.4952	0.0102	1.5417	0.0363
0.6	1.4903	1.4861	0.0042	1.4955	0.0053	1.5419	0.0516
0.7	1.4737	1.4658	0.0079	1.4958	0.0222	1.5421	0.0684
0.8	1.4391	1.4399	0.0007	1.4962	0.0571	1.5423	0.1032
0.9	1.4036	1.4029	0.0007	1.4965	0.0929	1.5425	0.1389
MAE		0.0050		0.0154		0.0363	
RMSE		0.0027		0.0061		0.0178	

Table 5: Comparing the Effect of Changing the Exdividend Date on Pricing Barrier Calls with Single Discrete Dividend
All numerical settings are the same as those settings in Table 2, except that the initial stock price is 50 and the exdividend date is listed in the first column.

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$S(0)$	Benchmark	Ours	error	ContDiv	error	Model1	error
46	0.9156	0.9122	0.0034	0.9003	0.0153	0.9493	0.0338
48	1.0033	1.0028	0.0005	0.9964	0.0069	1.0619	0.0586
50	1.0538	1.0493	0.0045	1.0481	0.0058	1.1322	0.0783
52	1.0484	1.0438	0.0046	1.0479	0.0005	1.1519	0.1035
54	0.9880	0.9843	0.0037	0.9934	0.0055	1.1178	0.1298
56	0.8771	0.8737	0.0035	0.8872	0.0101	1.0316	0.1545
58	0.7241	0.7192	0.0049	0.7357	0.0116	0.8990	0.1749
60	0.5364	0.5315	0.0049	0.5485	0.0122	0.7287	0.1924
62	0.3249	0.3233	0.0016	0.3371	0.0122	0.5317	0.2068
64	0.1104	0.1077	0.0032	0.1131	0.0027	0.3192	0.2088
MAE		0.0049		0.0153		0.2088	
RMSE		0.0038		0.0094		0.1470	

Table 6: Comparing the Effect of Changing Initial Stock Prices on Pricing Barrier Calls with Two Dividends
All settings are the same as the settings in Table 2 except that the underlying stock is assumed to pay 1 dollar dividend at year 0.5 and year 1, and the time to maturity is 1.5 years.

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$c_1 = c_2$	Benchmark	Ours	error	ContDiv	error	Model1	error
0.3	1.1305	1.1238	0.0067	1.1232	0.0073	1.1514	0.0210
0.6	1.0948	1.0933	0.0015	1.0923	0.0025	1.1462	0.0514
0.9	1.0585	1.0606	0.0021	1.0594	0.0010	1.1364	0.0780
1.2	1.0279	1.0269	0.0010	1.0248	0.0031	1.1222	0.0943
1.5	0.9864	0.9897	0.0033	0.9885	0.0021	1.1038	0.1174
1.8	0.9552	0.9535	0.0018	0.9508	0.0045	1.0812	0.1260
2.1	0.9147	0.9156	0.0009	0.9118	0.0030	1.0548	0.1401
2.4	0.8769	0.8772	0.0003	0.8715	0.0055	1.0247	0.1478
MAE		0.0068		0.0073		0.1478	
RMSE		0.0029		0.0041		0.1056	

Table 7: Comparing the Effect of Changing the Amounts of Dividends on Pricing Barrier Calls with Two Dividends
All settings are the same as the settings in Table 6, except that the initial stock price is 50, the underlying stock is assumed to pay dividend c_1 at year 0.5 and dividend c_2 at year 1. The amount of dividend payout is listed in the first column.

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A Reexpress the Integration of Exponential Functions in terms of the CDF of Multi-Variate Normal Distribution

If the price of the underlying asset is assumed to follow the lognormal diffusion process, most option pricing formulae, including the pricing formulae in this paper, can be expressed in terms of multiple integrations of an exponential function, where the exponent term is a quadratic function of integrators x_1, x_2, $\cdots$. The integration problem can be numerically solved by reexpressing the formulae in terms of CDF of Multi-Variate Normal Distribution, which can be efficiently solved by accurate numerical approximation methods (see Hull, 2003). These numerical methods are provided by mathematical softwares, like Matlab and Mathemaica. Take the simplest case– the single integral as an example. Under the premise

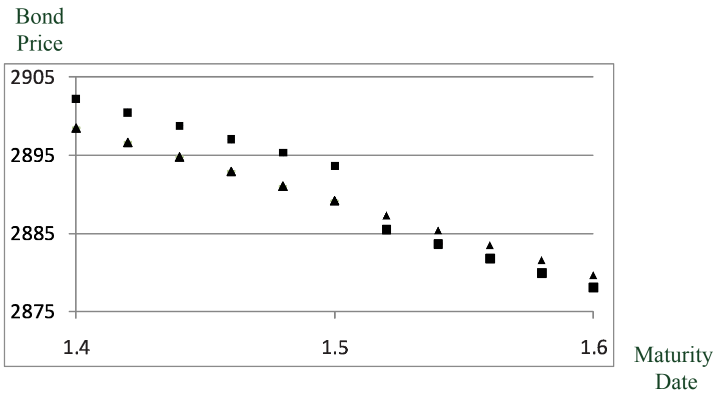

Figure 3: **Evaluating Vulnerable Bonds.** The x-axis denotes the bond maturity and the y-axis denotes the vulnerable bond price. The pricing results of our formulae are marked by solid squares. The pricing results by heuristically assuming that the firm pays a continuously payout are marked by solid triangles.

$a_2 < 0$[7], the integral $\int_{-\infty}^{l} e^{a_2x^2+a_1x+a_0}dx$ can be rewritten as

$$\begin{aligned} & \int_{-\infty}^{l} e^{a_2x^2+a_1x+a_0}dx \\ = & \int_{-\infty}^{l} e^{-\frac{1}{2}\left(\frac{x-\mathbf{m}}{\mathbf{s}}\right)^2-\frac{a_1^2-4a_0a_2}{4a_2}}dx \\ = & \sqrt{2\pi}e^{-\frac{a_1^2-4a_0a_2}{4a_2}}\int_{-\infty}^{\frac{l-\mathbf{m}}{\mathbf{s}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\mathbf{s}dy \\ = & \sqrt{\frac{\pi}{-a_2}}e^{-\frac{a_1^2-4a_0a_2}{4a_2}}N\left(\frac{l-\mathbf{m}}{\mathbf{s}}\right), \end{aligned} \tag{46}$$

where $\mathbf{m} = -\frac{a_1}{2a_2}$ and $\mathbf{s} = \frac{1}{\sqrt{-2a_2}}$ in the second equality, change of variables is applied with the equation $y = \frac{x-\mathbf{m}}{\mathbf{s}}$ in the third equality, $N(\cdot)$ in the last equality denotes the CDF of a univariate standard normal distribution.

However, the integration for multivariate case is not straightforward. To address this problem, we derive a general formula for the multivariate integration with n integrators: $x_1, x_2, \cdots, x_n$. Assume that $\mathbf{x} \equiv (x_1, x_2, \cdots, x_n)^T$ is a column vector with n variables. Then the general integral formula is derived in Theorem 3.1 and the proof is given as follows:

Proof.

To express the integral in Eq. (22) in terms of a CDF of a standard normal distribution, the exponent term $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C}$ should be expressed in terms of the exponent term of a standard normal distribution as follows:

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} + \mathbf{C} = -\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y} + \mathbf{C}', \tag{47}$$

[7] The definite integral is infinite if $a_2 > 0$.

where the scalar $\mathbf{C}^{'}$ does not depend on $\mathbf{x}$. This can be achieved by applying the completing the square technique on $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x}$ as discussed in the following lemma:

Lemma A.1 *Under the premises that* $\mathbf{A}$ *is a symmetric invertible* $n \times n$ *matrix, and* $\mathbf{x}, \mathbf{B}$ *are both* $n \times 1$ *vectors, we have*

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} = \left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right)^T \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right) - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}. \tag{48}$$

Proof. By expanding the right-hand side of Eq. (48), we have

$$\begin{aligned}
& \left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right)^T \mathbf{A}\left(\mathbf{x} + \frac{1}{2}\mathbf{A}^{-1}\mathbf{B}\right) - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \\
= & \ \mathbf{x}^T\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{B}^T\left(\mathbf{A}^{-1}\right)^T\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{A}^{-1}\mathbf{B} + \frac{1}{4}\mathbf{B}^T\left(\mathbf{A}^{-1}\right)^T\mathbf{A}\mathbf{A}^{-1}\mathbf{B} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \quad (49) \\
= & \ \mathbf{x}^T\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{B}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{B} + \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \quad (50) \\
= & \ \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{B}^T\mathbf{x} = \text{the left-hand side of Eq. (48)}, \quad (51)
\end{aligned}$$

where the equation $\left(\mathbf{A}^{-1}\right)^T = \mathbf{A}^{-1}$ due to the symmetry of $\mathbf{A}$ is substituted into Eq. (49). Since $\mathbf{B}^T\mathbf{x}$ is a scalar, we have $\mathbf{B}^T\mathbf{x} = \left(\mathbf{B}^T\mathbf{x}\right)^T = \mathbf{x}^T\mathbf{B}$. Eq. (50) is obtained by substituting the aforementioned equalities into Eq. (49). □

With Lemma A.1, we obtain

$$\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B} + \mathbf{C} = (\mathbf{x} - \mathbf{m})^T\mathbf{A}(\mathbf{x} - \mathbf{m}) + \mathbf{C}^{'}, \tag{52}$$

where $\mathbf{m} \equiv -\frac{1}{2}\mathbf{A}^{-1}\mathbf{B}$, and $\mathbf{C}^{'} \equiv \mathbf{C} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. By equating the right-hand sides of Eq. (47) and Eq. (52), we have

$$(\mathbf{x} - \mathbf{m})^T\mathbf{A}(\mathbf{x} - \mathbf{m}) + \mathbf{C}^{'} = -\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y} + \mathbf{C}^{'}. \tag{53}$$

It can be observed that $\mathbf{y}$ should have the form

$$\mathbf{y} = \mathbf{S}^{-1}(\mathbf{x} - \mathbf{m}), \tag{54}$$

where $\mathbf{S}$ denotes a diagonal matrix. To solve $\mathbf{S}$, we first subtract $\mathbf{C}^{'}$ from both sides of Eq. (53) to yield

$$\begin{aligned}
(\mathbf{x} - \mathbf{m})^T\mathbf{A}(\mathbf{x} - \mathbf{m}) &= -\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y} \quad (55) \\
&= -\frac{1}{2}\left(\mathbf{S}^{-1}(\mathbf{x} - \mathbf{m})\right)^T\Sigma^{-1}\left(\mathbf{S}^{-1}(\mathbf{x} - \mathbf{m})\right) \quad (56) \\
&= -\frac{1}{2}(\mathbf{x} - \mathbf{m})^T\mathbf{S}^{-1}\Sigma^{-1}\mathbf{S}^{-1}(\mathbf{x} - \mathbf{m}) \\
&= -\frac{1}{2}(\mathbf{x} - \mathbf{m})^T(\mathbf{S}\Sigma\mathbf{S})^{-1}(\mathbf{x} - \mathbf{m}), \quad (57)
\end{aligned}$$

where Eq. (54) is substituted into the right-hand side of Eq. (55), $(\mathbf{S}^{-1})^T = \mathbf{S}^{-1}$ due to the symmetry of $\mathbf{S}$ is substituted into Eq. (56). By comparing the left-hand side of Eq. (55) and Eq. (57), we have $-\frac{1}{2}(\mathbf{S}\Sigma\mathbf{S})^{-1} = \mathbf{A}$, which can be rewritten as $\mathbf{S}\Sigma\mathbf{S} = (-2\mathbf{A})^{-1}$. Recall that $\mathbf{S}$ is a diagonal matrix. All diagonal elements of Σ are 1 since Σ is a covariance matrix of multivariate standard normal random variables. Thus we have $(\mathbf{S}\Sigma\mathbf{S})_{i,i} = \mathbf{S}_{i,i}^2$, which leads us to obtain

$$\mathbf{S}_{i,j} \equiv \begin{cases} \sqrt{((-2\mathbf{A})^{-1})_{i,i}} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

and $\Sigma \equiv (-2\mathbf{SAS})^{-1}$.

Now we can express Eq. (22) in terms of $\mathbf{C}'$, $\mathbf{m}$, $\mathbf{S}$ and Σ defined above. By applying the change of variable Eq. (54), Eq. (22) can be rewritten as

$$\begin{aligned} & \int_{x_n=-\infty}^{x_n=p_n} \int_{x_{n-1}=-\infty}^{x_{n-1}=p_{n-1}} \cdots \int_{x_1=-\infty}^{x_1=p_1} e^{\mathbf{x}^T\mathbf{A}\mathbf{x}+\mathbf{B}^T\mathbf{x}+\mathbf{C}} d\mathbf{x} \\ = & \int_{x_n=-\infty}^{x_n=p_n} \int_{x_{n-1}=-\infty}^{x_{n-1}=p_{n-1}} \cdots \int_{x_1=-\infty}^{x_1=p_1} e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}+\mathbf{C}'} \left|\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\right| d\mathbf{y}. \end{aligned} \tag{58}$$

Since the elements in vector $\mathbf{y}$ can be represented as $\left(\frac{x_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}, \frac{x_2-\mathbf{m}_2}{\mathbf{S}_{2,2}}, \cdots, \frac{x_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}\right)^T$, the Jacobian determinant can be straightforwardly computed to get $\left|\frac{\partial \mathbf{x}}{\partial \mathbf{y}}\right| = \prod_{i=1}^n \mathbf{S}_{i,i} = |\mathbf{S}|$. Thus, Eq. (58) can be further rewritten as the following closed form formula:

$$\begin{aligned} & e^{\mathbf{C}'}|\mathbf{S}| \int_{-\infty}^{\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}} \int_{-\infty}^{\frac{p_{n-1}-\mathbf{m}_{n-1}}{\mathbf{S}_{n-1,n-1}}} \cdots \int_{-\infty}^{\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}} e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}} d\mathbf{y} \\ = & e^{\mathbf{C}'}|\mathbf{S}|\sqrt{|\Sigma|}\sqrt{(2\pi)^n} \int_{-\infty}^{\frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}} \int_{-\infty}^{\frac{p_{n-1}-\mathbf{m}_{n-1}}{\mathbf{S}_{n-1,n-1}}} \cdots \int_{-\infty}^{\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}} \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} e^{-\frac{1}{2}\mathbf{y}^T\Sigma^{-1}\mathbf{y}} d\mathbf{y} \quad (59) \\ = & e^{\mathbf{C}'}\sqrt{\frac{\pi^n}{|-\mathbf{A}|}} F_{Y_1,Y_2,\cdots,Y_n}\left(\frac{p_1-\mathbf{m}_1}{\mathbf{S}_{1,1}}, \frac{p_2-\mathbf{m}_2}{\mathbf{S}_{2,2}}, \cdots, \frac{p_n-\mathbf{m}_n}{\mathbf{S}_{n,n}}, \Sigma\right), \end{aligned}$$

where $|\mathbf{S}|\sqrt{|\Sigma|} = \sqrt{|\mathbf{S}\Sigma\mathbf{S}|} = \sqrt{|-2\mathbf{A}|^{-1}}$, and $|-2\mathbf{A}| = 2^n|-\mathbf{A}|$ are substituted into Eq. (59). Note that the univariate integration in Eq. (46) is a special case of this theorem. □

B Proof of Corollary 3.1

Corollary 4.1 can be derived from Theorem 3.1 by setting $n = 2$ as follows:

First, to make $a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$ equal $\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{x}^T\mathbf{B} + \mathbf{C}$, we set

$$\mathbf{A} \equiv \begin{pmatrix} a_1 & \frac{a_2}{2} \\ \frac{a_2}{2} & a_3 \end{pmatrix}, \mathbf{B} \equiv \begin{pmatrix} a_4 \\ a_5 \end{pmatrix}, \mathbf{C} \equiv a_6.$$

By substituting above equations into Theorem 3.1, we have

$$\begin{aligned}
\mathbf{m} &= -\frac{1}{2}\mathbf{A}^{-1}\mathbf{B} = -\frac{1}{4a_1a_3 - a_2^2}\begin{pmatrix} 2a_3a_4 - a_2a_5 \\ 2a_1a_5 - a_2a_4 \end{pmatrix}, \\
\mathbf{C}' &= \mathbf{C} - \frac{1}{4}\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} = a_6 + \frac{a_2a_4a_5 - a_3a_4^2 - a_1a_5^2}{4a_1a_3 - a_2^2}, \\
\mathbf{S}_{1,1} &= \sqrt{((-2\mathbf{A})^{-1})_{1,1}} = \sqrt{\frac{-2a_3}{4a_1a_3 - a_2^2}}, \\
\mathbf{S}_{2,2} &= \sqrt{((-2\mathbf{A})^{-1})_{2,2}} = \sqrt{\frac{-2a_1}{4a_1a_3 - a_2^2}}, \\
\Sigma &= (-2\mathbf{SAS})^{-1} = \begin{pmatrix} 1 & \frac{a_2}{2\sqrt{a_1a_3}} \\ \frac{a_2}{2\sqrt{a_1a_3}} & 1 \end{pmatrix}, \\
|-\mathbf{A}| &= \frac{4a_1a_3 - a_2^2}{4}.
\end{aligned}$$

By substituting the above into Eq. (23), we obtain

$$\begin{aligned}
&\int_{-\infty}^{q}\int_{-\infty}^{p} e^{a_1x^2+a_2xy+a_3y^2+a_4x+a_5y+a_6}dxdy \\
= \;& e^{\mathbf{C}'}\sqrt{\frac{\pi^2}{|-\mathbf{A}|}}F_{Y_1,Y_2}\left(\frac{p - \mathbf{m}(1)}{\mathbf{S}_{1,1}}, \frac{q - \mathbf{m}(2)}{\mathbf{S}_{2,2}}, \Sigma\right) \\
= \;& \frac{2\pi}{\sqrt{\Delta}}\exp\left(a_6 + \frac{a_2a_4a_5 - a_3a_4^2 - a_1a_5^2}{\Delta}\right)F_{Y_1,Y_2}\left(\frac{\sqrt{\Delta}p + \frac{2a_3a_4 - a_2a_5}{\sqrt{\Delta}}}{\sqrt{-2a_3}}, \frac{\sqrt{\Delta}q + \frac{2a_1a_5 - a_2a_4}{\sqrt{\Delta}}}{\sqrt{-2a_1}}, \Sigma\right),
\end{aligned}$$

where $\Delta = 4a_1a_3 - a_2^2$.

C Coefficients of the Exponential Terms of $J(i)$

The coefficients $a_1(i)$, $a_2(i)$, $a_3(i)$, $a_4(i)$, $a_5(i)$, and $a_6(i)$ are defined by the following formulae.

$$
\begin{aligned}
a_1(i) &= -\frac{1}{2\left(T-t_2\right)}, \\
a_2(i) &= -\left(\frac{k_2^2}{2\left(T-t_2\right)}+\frac{1}{2(t_2-t_1)}\right), \\
a_3(i) &= -\left(\frac{k_1^2}{2\left(t_2-t_1\right)}+\frac{1}{2t_1}\right), \\
a_6(i) &= 0, \\
a_4(i) &= \begin{cases} \frac{k_2}{T-t_2} & \text{if } i=1,2,5,6,9,10,13,14 \\ -\frac{k_2}{T-t_2} & \text{otherwise} \end{cases}, \\
a_5(i) &= \begin{cases} \frac{k_2}{t_2-t_1} & \text{if } i=1,2,3,4,9,10,11,12 \\ -\frac{k_2}{t_2-t_1} & \text{otherwise} \end{cases},
\end{aligned}
$$

The rest parameters are given by the following table:

i	$a_7(i)$	$a_8(i)$	$a_9(i)$	$a_{10}(i)$
1	α	$\alpha-\alpha k_2$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
2	α	$\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
3	$\frac{2b^{''}}{T-t_2}+\alpha$	$\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\alpha-\alpha k_1$	$-\frac{4b^{''2}+T^2\alpha^2-T\alpha^2t_2}{2T-2t_2}$
4	$\frac{2b^{''}}{T-t_2}+\alpha$	$\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$\frac{4(t_2-T)b^2+t_1\left(-4b^{''2}-T^2\alpha^2+T\alpha^2t_2\right)}{2t_1(T-t_2)}$
5	α	$\frac{2b^{'}}{t_2-t_1}+\alpha-\alpha k_2$	$\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{4b^{'2}-T\alpha^2t_1+T\alpha^2t_2}{2t_1-2t_2}$
6	α	$\frac{2b^{'}}{t_2-t_1}+\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{4t_2b^2-T\alpha^2t_1^2+t_1\left(-4b^2+4b^{'2}+T\alpha^2t_2\right)}{2t_1(t_1-t_2)}$
7	$\frac{2b^{''}}{T-t_2}+\alpha$	$\frac{2b^{'}}{t_2-t_1}+\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{2b^{'2}}{t_1-t_2}-\frac{T\alpha^2}{2}-\frac{2b^{''2}}{T-t_2}$
8	$\frac{2b^{''}}{T-t_2}+\alpha$	$\frac{2b^{'}}{t_2-t_1}+\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b^{''2}}{T-t_2}+\frac{2b^{'2}}{t_1-t_2}$
9	$\alpha+\sigma$	$\alpha-\alpha k_2$	$\alpha-\alpha k_1$	$-\frac{T\alpha^2}{2}$
10	$\alpha+\sigma$	$\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}$
11	$\frac{2b^{''}}{T-t_2}+\alpha+\sigma$	$\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\alpha-\alpha k_1$	$-\frac{4b^{''2}+T^2\alpha^2-T\alpha^2t_2}{2T-2t_2}$
12	$\frac{2b^{''}}{T-t_2}+\alpha+\sigma$	$\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha-\alpha k_1$	$\frac{4(t_2-T)b^2+t_1\left(-4b^{''2}-T^2\alpha^2+T\alpha^2t_2\right)}{2t_1(T-t_2)}$
13	$\alpha+\sigma$	$\frac{2b^{'}}{t_2-t_1}+\alpha-\alpha k_2$	$\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{4b^{'2}-T\alpha^2t_1+T\alpha^2t_2}{2t_1-2t_2}$
14	$\alpha+\sigma$	$\frac{2b^{'}}{t_2-t_1}+\alpha-\alpha k_2$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{4t_2b^2-T\alpha^2t_1^2+t_1\left(-4b^2+4b^{'2}+T\alpha^2t_2\right)}{2t_1(t_1-t_2)}$
15	$\frac{2b^{''}}{T-t_2}+\alpha+\sigma$	$\frac{2b^{'}}{t_2-t_1}+\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$\frac{2b^{'2}}{t_1-t_2}-\frac{T\alpha^2}{2}-\frac{2b^{''2}}{T-t_2}$
16	$\frac{2b^{''}}{T-t_2}+\alpha+\sigma$	$\frac{2b^{'}}{t_2-t_1}+\alpha+k_2\left(\frac{2b^{''}}{T-t_2}-\alpha\right)$	$\frac{2b}{t_1}+\alpha+k_1\left(\frac{2b^{'}}{t_2-t_1}-\alpha\right)$	$-\frac{2b^2}{t_1}-\frac{T\alpha^2}{2}-\frac{2b^{''}S^2}{T-t_2}+\frac{2b^{'2}}{t_1-t_2}$